THE NUMERICAL METHODS FOR SOLVING NONLINEAR INTEGRAL EQUATIONS

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Abstract:
Volterra integral equations are a special type of integrative equations; they are divided into two categories referred to as the first and second type. This thesis will deal with the second type which has wide range of the applications in physics and engineering problems. The aim of this paper is to compare between analytical solution and numerical solution to solve Integral Equations. Numerical examples are presented to illustrate the applications of this methods and to compare the computed results with other numerical methods for analytical solutions. Finally by comparison of numerical results, Simplicity and efficiency of this method be shown.

Keywords: *Nonlinear Integral Equations, Volterra Integral Equation, Fredholm Integral Equation, Numerical, Methods.*

1.INTRODUCTION:

We studied the nonlinear Fredholm integral equations of the first and the second kinds. We pointed out earlier that nonlinear integral equations need a considerable amount of work. However, with the recent developed methods we can minimize significantly this cumbersome work. It is therefore useful to present some reliable and powerful techniques that will make the study of nonlinear integral equations successful and valuable. In general, the solution of the nonlinear integral equations is not in general unique as we studied in the case of the nonlinear Fredholm integral equations. However, the existence of a unique solution of nonlinear integral equations with specific conditions is possible but cannot be assumed general. This will be illustrated by the forthcoming examples. Accordingly, our emphasis will be on introducing reliable and easily calculable techniques for solving specific cases of nonlinear Volterra integral equations. As indicated in Chapter 1, integral equations of the form.

$$u(x) = f(x) + \lambda \int_a^b K(x,t)F\big(u(t)\big)dt \qquad (1)$$

And

$$u(x) = f(x) + \lambda \int_0^x K(x,t)F\big(u(t)\big)dt \qquad (2)$$

are called *nonlinear* Fredholm integral equations and *nonlinear* Volterra integral equations respectively. The function $F\big(u(x)\big)$ is nonlinear in $u(x)$ such as $u^2(x)$, $u^3(x)$, $e^{u(x)}$, $\sin u(x)$, and many others, and λ is a parameter. In this text, we will concern ourselves to the case where $F(u(t)) = un(t)$, $n \geq 2$, whereas other nonlinear integral equations that involve other forms of nonlinearity of $F(u(x))$ can be handled in a very similar way. The following are examples of the *nonlinear* Volterra integral equations of the second kind.

$$u(x) = x - \frac{1}{4}x^4 + \int_0^x t\,u^2(t)dt. \qquad (3)$$

$$u(x) = 2x - \frac{1}{6}x^5 - \int_0^x t\,u^3(t)dt. \qquad (4)$$

Moreover, the nonlinear Volterra integral equations of the first kind are of the form

$$x = \int_0^x t\,u^2(t)dt\,, \qquad (5)$$

$$2x = -\int_0^x t^2\,u^4(t)dt\,, \qquad (6)$$

2.Nonlinear Fredholm Integral Equations:

We have been mainly concerned with studying different methods for solving linear integral equations of the first and the second kind. We pointed out earlier that nonlinear integral equations yield a considerable amount of difficulties. However, with the recent methods developed, it seems reasonable to present some reliable and powerful techniques that will make the study of specific cases of nonlinear integral equations successful and valuable. In general, the solution of the nonlinear integral equation is not unique. However, the existence of a solution of nonlinear integral equations with specific conditions is possible. Because we will concern ourselves with nonlinear Fredholm integral equations that will give solutions, therefore we will not discuss in this text the theorem of existence of solutions of nonlinear equations. For more information about the conditions that are necessary for the existence of solutions for nonlinear equations, the reader is advised to look in other texts such as [19] and [44].

For solving specific cases of nonlinear Fredholm integral equations.
given $F(u(t))$ a nonlinear function in $u(t)$, integral equations of the form

$$u(x) = f(x) + \lambda \int_a^b K(x,t)F\big(u(t)\big)dt \qquad (7)$$

And

$$u(x) = f(x) + \lambda \int_0^x K(x,t)F\big(u(t)\big)dt \qquad (8)$$

are called nonlinear Fredholm integral equations and *nonlinear* Volterra integral equations respectively. The function $F(u(t))$ is nonlinear in $u(t)$ such as $u^2(t)$, $u^3(t)$, $e^{u(t)}$, and λ is a parameter. However, we will restrict our discussion to the case where $F\big(u(t)\big) = u^n(t), n \geq 2$,whereas other nonlinear integral equations that involve nonlinear terms other than $un(t)$ can be handled in a very similar manner. The following are examples of nonlinear Fredholm integral equations:

$$u(x) = 1 + \lambda \int_0^1 u^2(t)dt\,, \qquad (9)$$

$$u(x) = x + \int_0^1 xtu^3(t)dt\,. \qquad (10)$$

3. Solving Nonlinear Integral Equations Using Numerical Methods:

i.Nonlinear Volterra Integral Equations of the Second Kind

We begin our study on nonlinear Volterra integral equations of the second kind given by

$$u(x) = f(x) + \int_0^x K(x,t)F(u(t))dt\,. \qquad (11)$$

where the kernel $K(x,t)$ and the function $f(x)$ are given real-valued functions, $F(u(x))$ and is a nonlinear function of $u(x)$ such as $u^3(x)$, $\cos(u(x))$, and $e^{u(x)}$. The unknown function $u(x)$, that will be determined, occurs inside and outside the integral sign.

The nonlinear Volterra equation (11) will be handled by using three distinct methods. The three methods are the successive approximations method, the series solution method, and the Adomian decomposition method (ADM). The latter will be combined with the modified decomposition method and the noise terms phenomenon.

a. The Successive Approximations Method:

The successive approximations method [7], or the Picard iteration method was used before. This method solves any problem by finding successive approximations to the solution by starting with an initial guess, called the zeroth approximation. As will be seen later, the zeroth approximation is any selective real-valued function that will be used in a recurrence relation to determine the other approximations.

Given the nonlinear Volterra integral equation of the second kind

$$u(x) = f(x) + \int_0^x K(x,t)F(u(t))dt\,. \qquad (12)$$

where $u(x)$ is the unknown function to be determined and $K(x,t)$ is the kernel. The successive approximations method introduces the recurrence relation

$$u_{n+1}(x) = f(x) + \int_0^x K(x,t)F(u_n(t))dt, \;\; n \geq 0\,. \qquad (13)$$

where the zeroth approximation $u_0(x)$ can be any selective real valued function. We always start with an initial guess for $u_0(x)$, mostly we select 0, 1, or x for $u_0(x)$. Using this selection of $u_0(x)$ into (4.8), several successive approximations u_k , $k \geq 1$ will be determined as

$$\begin{aligned}
u_1(x) &= f(x) + \int_0^x K(x,t)\,F\big(u_0(t)\big)dt \;, \\
u_2(x) &= f(x) + \int_0^x K(x,t)\,F\big(u_1(t)\big)dt \;, \\
u_3(x) &= f(x) + \int_0^x K(x,t)\,F\big(u_2(t)\big)dt \\
&\vdots \\
u_{n+1}(x) &= f(x) + \int_0^x K(x,t)\,F\big(u_n(t)\big)dt \;.
\end{aligned} \qquad (14)$$

Consequently, the solution $u(x)$ is obtained by using

$$u(x) = \lim_{n\to\infty} u_{n+1}(x) \;. \qquad (15)$$

The question of convergence of $u_{n+1}(x)$ was examined in Chapter 3. The successive approximations method, or the Picard iteration method will be illustrated by the following examples.

Example (3.1):

Use the successive approximations method to solve the nonlinear Volterra integral equation

$$u(x) = e^x + \frac{1}{3}x(1-e^{3x}) + \int_0^x xu^3(t)dt. \quad (16)$$

For the zeroth approximation $u_0(x)$, we can select

$$u_0(x) = 1 \qquad (17)$$

The method of successive approximations admits the use of the iteration formula

$$u(x) = e^x + \frac{1}{3}x(1-e^{3x}) + \int_0^x xu^3(t)dt \;, \; n \geq 1 \qquad (18)$$

Substituting (12) into (13) we obtain the approximations

$$\begin{aligned}
u_0(x) &= 1 \\
u_1(x) &= e^x + \frac{1}{3}x(1-e^{3x}) + \int_0^x x{u_0}^3(t)dt \\
&= 1 + x + \frac{1}{2!}x^2 - \frac{4}{3}x^3 - \frac{35}{24}x^3 - \frac{67}{60}x^5 + \cdots, \\
u_2(x) &= e^x + \frac{1}{3}x(1-e^{3x}) + \int_0^x x{u_1}^3(t)dt \\
&= 1 + x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 - \frac{1}{4!}x^4 - \frac{67}{60}x^5 + \cdots, \\
u_3(x) &= e^x + \frac{1}{3}x(1-e^{3x}) + \int_0^x x{u_2}^3(t)dt \\
&= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \cdots,
\end{aligned} \qquad (19)$$

and so on. Consequently, the solution $u(x)$ of (11) is given by

$$u(x) = \lim_{n\to\infty} u_n(x) \qquad (20)$$

b. The Series Solution Method:

The series solution method was applied to handle linear Volterra and Fredholm integral equations. The series solution method will be applied in a similar manner to handle the nonlinear Volterra integral equations.
Recall that the generic form of Taylor series at $x = 0$ can be written as

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \qquad (21)$$

We will assume that the solution $u(x)$ of the nonlinear Volterra integral equations

$$u(x) = f(x) + \int_0^x K(x,t)\, F(u(t)dt)\,, \qquad (22)$$

is analytic, and therefore possesses a Taylor series of the form where the coefficient a_n will be determined recurrently. By substituting it gives

$$\sum_{n=0}^{\infty} a_n x^n = T\big(f(x)\big) + \int_0^x K(x,t)\, F(\sum_{n=0}^{\infty} a_n t^n \;\; dt) \qquad (23)$$

or for simplicity we use

$$a_0 + a_1 x + a_2 x^2 + \cdots = T\big(f(x)\big) + \int_0^x K(x,t)\, \big(F(a_0 + a_1 t + a_2 t^2)\big)dt, \quad (24)$$

where $T\big(f(x)\big)$ is the Taylor series for $f(x)$. by integrating the nonlinear term $T\big(f(x)\big)$, terms of the form t^n, $n \geq 0$ will be integrated. Notice that because we are seeking series solution, then if $f(x)$ includes elementary functions such as trigonometric functions, exponential functions, etc., then Taylor expansions for functions involved in $f(x)$ should be used.

Example (3.2):

Solve the following nonlinear Volterra integral equation by using the series solution method

$$u(x) = 1 + x + \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{12}x^4 + \int_0^x (x-t)u^2(t)dt\,. \quad (25)$$

Using the series form (4.31) into both sides of (4.35) gives

$$a_0 + a_1 x + a_2 x^2 + \cdots = 1 + x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{12}x^4 + \int_0^x (x-t)(a_0 + a_1 t + a_2 t^2 + a_3 t^3)dt\,. \qquad (26)$$

where by integrating the integral at the right side, and collecting like powers of x we obtain

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = 1 + x + \tfrac{1}{2}(a_0 - 1)x^2 + \tfrac{1}{3}(a_0 a_1 - 1)x^3 + ({a_1}^2 + 2a_0 a_2 - 1)x^4 + \cdots \qquad (27)$$

Equating the coefficients of like powers of x in both sides yields

$$a_0 = 1\,,\;\; a_1 = 1\,, \qquad a_n = 0\,, \quad \text{for} \;\; n \geq 2 \qquad (28)$$

The exact solution is given by

$$u(x) = 1 + x\,. \qquad (29)$$

4. Nonlinear Volterra Integral Equations of the First Kind:

The standard form of the nonlinear Volterra integral equation of the first kind is given by

$$f(x) = \int_0^x K(x,t)\, F\big(u(t)\big)\,, \qquad (30)$$

where the kernel $K(x,t)$ and the function $f(x)$ are given real-valued functions, and $F\big(u(t)\big)$ is a nonlinear function of $u(x)$ Recall that the unknown function $u(x)$ occurs only inside the integral sign for the Volterra integral equation of the first kind. The linear Volterra integral equation of the first kind is presented where three main methods were used for handling this kind of equation.

To determine a solution for the nonlinear Volterra integral equation of the first kind we first convert it to a linear Volterra integral equation of the first kind of the form

$$f(x) = \int_0^x K(x,t)\, v(t)\,, \qquad (31)$$

by using the transformation

$$v(x) = F\big(u(t)\big), \qquad (32)$$

This in turn means that

$$u(x) = F^{-1}\big(v(x)\big)\ . \qquad (33)$$

It is worth noting that the Volterra integral equation of the first kind can be solved by any method that was shown.

a. The Laplace Transform Method:

The Laplace transform method is a powerful technique that we used before for solving Volterra integral equations of the first and the second kinds. We assume that the kernel $K(x,t)$ is a difference kernel. Taking the Laplace transforms of both sides of (4.82) gives

$$L\{f(x)\} = L\{k(x-t)\} \times L\{v(x)\}, \qquad (34)$$

So that

$$v(s) = \frac{F(s)}{K(s)}\ , \qquad (35)$$

Where

$$F(s) = L\{f(x)\}, K(s) = L\{K(x)\}, v(s) = L\{v(x)\}, \qquad (36)$$

Taking the inverse Laplace transform of both sides of (4.86) gives $v(x)$. The Solution $u(x)$ is obtained. It is obvious that the Laplace Transform method works effectively provided that

$$\lim_{s\to\infty} \frac{F(s)}{K(s)} = 0\ , \qquad (37)$$

The Laplace transform method will be used for studying the following nonlinear Volterra integral equations of the first kind.

Example (4.1):

Solve the nonlinear Volterra integral equation of the first kind by using the Laplace transform method

$$\frac{1}{4}e^{2x} - \frac{1}{2}x - \frac{1}{4} = \int_0^x (x-t)\,u^2(t)dt.$$

We first set

$v(x) = u^2(x), u(x) = \pm\sqrt{v(x)}$ to carry out (33) into

$$\frac{1}{4}e^{2x} - \frac{1}{2}x - \frac{1}{4} = \int_0^x (x-t)\,v(t)dt. \qquad (38)$$

Taking the Laplace transform of both sides of (34) yields

$$\frac{1}{4(s-2)} - \frac{1}{2s^2} - \frac{1}{4s} = \frac{1}{s^2}v(s), \qquad (39)$$

or equivalently

$$v(s) = \frac{1}{(s-2)}\ , \qquad (40)$$

Where

$$v(s) = L\{v(x)\}\ , \qquad (41)$$

Taking the inverse Laplace transform of both sides of (35) gives

$$v(x) = e^{2x}\ . \qquad (42)$$

The exact solutions are therefore given by

$$u(x) = \pm e^x. \qquad (43)$$

It is worth noting that two solutions were obtained because Eq. (31) is a nonlinear equation, and the solution may not be unique.

b. Conversion to a Volterra Equation of the Second Kind:

Consider the nonlinear Volterra integral equation of the first kind

$$f(x) = \int_0^x K(x,t)F(u,t)\,dt, \qquad (44)$$

Where the kernel $k(x,t)$ and the function $f(x)$ are given rela-valued functions, and $u(x)$ is the to be determined. In a manner parallel to our discussion before, we convert to a linear volterra integral equation of first kind of the form

$$f(x) = \int_0^x K(x,t)v(t)\,dt, \qquad (45)$$

By using the transformation

$$v(x) = F\big(u(x)\big). \quad (46)$$

This in turn means that

$$u(x) = F^{-1}(v(x)). \quad (47)$$

We next convert the linear voltera integral equations of the first kind to voltera integral equations of the second kind. The conversion technique works effectively only if $k(x,x) \neq 0$. Differentiating both sides with respect to x, and using Leibnitz rule, we find

$$f`(x) = k(x,x)v(x) + \int_0^x k_x(x,y)v(t)\,dt \quad (48)$$

Solving for $v(x)$, provided that $(x,x) \neq 0$. we obtain the volterra integral equation of the second kind given by

$$v(x) = \frac{f`(x)}{k(x,x)} - \int_0^x \frac{1}{k(x,x)} k_x(x,y)v(t)\,dt \quad (49)$$

Having converted the Volterra integral equation of the first kind to the Volterra integral equation of the second kind, we then can use any method that was presented before. Because we solved the Volterra integral equations of the second kind by many methods, we will select distinct methods for solving the nonlinear Volterra integral equation of the first kind after converting it to a Volterra integral equation of the second kind

Example (4.2):
Convert the nonlinear Volterra integral equation of the first kind to the second kind and solve the resulting equation

$$\frac{1}{3}(e^x - e^{-2x}) = \int_0^x e^{x-t}\, u^2(t)\,dt. \quad (50)$$

We first set

$$v(x) = u^2(x), u(x) = \pm\sqrt{v(x)}\,, \quad (51)$$

To carry out

$$\frac{1}{3}(e^x - e^{-2x}) = \int_0^x e^{x-t}\, v(t)\,dt. \quad (52)$$

Differentiating both sides with respect to x by using Leibnitz rule we find the Volterra integral equation of the second kind

$$v(x) = \frac{1}{3}(e^x - e^{-2x}) - \int_0^x e^{x-t}\, v(t) \quad (53)$$

This can be solved by many methods as presented. We select the Laplace transform method to solve this equation. Taking Laplace transform of both sides yields

$$v(x) = \frac{1}{s+2} \quad (54)$$

Taking the inverse Laplace transform of both sides gives.

$$v(x) = e^{-2x} \quad (55)$$

The exact solutions are there for given by

$$v(x) = \pm e^{-x} \quad (56)$$

5.Nonlinear Fredholm Integral Equations of the Second Kind:
We begin our study on nonlinear Fredholm integral equations of the second kind of the form

$$u(x) = f(x) + \lambda \int_a^b K(x,t)\, F\big(u(t)\big)dt \quad (57)$$

where the kernel $K(x,t)$ and the function $f(x)$ are given real-valued functions, and $F(u(x))$ is a nonlinear function of $u(x)$. The unknown function $u(x)$. , that will be determined, occurs inside and outside the integral sign. In what follows we will employ four distinct methods, namely the direct computation method, the series solution method, the Adomian decomposition method, and the successive approximations method to handle the equation.

a. The Direct Computation Method
In this section, the direct computational method will be applied to solve the nonlinear Fredholm integral equations. The method was used before in Chapters 4 and 6. It approaches nonlinear Fredholm integral equations in a direct manner and gives the solution in an exact form and not in a series form. It is important to point out that this method will be applied for equations where the kernels are degenerate or separable of the form

$$K(x,t) = \sum_{k=1}^{n} g_k(x)\, h_k(t). \quad (58)$$

Examples of separable kernels are $x - t,\ xt^2, x^3 - t^3, xt^4 + x^4t$, etc.

The direct computation method can be applied as follows:
1.We first substitute (15.10) into the nonlinear Fredholm integral equation

$$u(x) = f(x) + \lambda \int_a^b K(x,t)\, F\big(u(t)\big)dt \qquad (59)$$

2. This substitution gives

$$u(x) = f(x) + \lambda g_1(x) \int_a^b h_1(t)\, F\big(u(t)\big)dt + \lambda g_2(x) \int_a^b h_2(t)\, F\big(u(t)\big)dt + \cdots + \lambda g_n(x) \int_a^b h_n(t)\, F\big(u(t)\big)dt. \qquad (60)$$

3. Each integral at the right side depends only on the variable t with constant limits of integration for t . This means that each integral is equivalent to a constant. Based on this, the equation becomes

$$u(x) = f(x) + \lambda\, \alpha_1 g_1(x) + \lambda\, \alpha_2 g_2(x) + \cdots + \lambda\, \alpha_n g_n(x) \qquad (61)$$

Where

$$\alpha_i = \int_a^b h_i(t)\, u(t)dt, \qquad 1 \le i \le n. \qquad (62)$$

Example (5.1):
Use the direct computation method to solve the nonlinear Fredholm integral equation

$$u(x) = a + \lambda \int_0^1 u^2(t)\, dt, \quad a > 0. \qquad (63)$$

The integral at the right side of (63) is equivalent to a constant because it depends only on a function of the variable *t* with constant limits of integration Consequently, we rewrite (63) as

$$u(x) = a + \lambda\alpha, \qquad (64)$$

where

$$\alpha = \int_0^1 u^2(t)\, dt. \qquad (65)$$

$$\alpha = \int_0^1 (a + \lambda\alpha)^2(t)\, dt. \qquad (66)$$

where by integrating the right side we find

$$\lambda^2 a^2 - (1 - 2\lambda a)\alpha + a^2 = 0 \qquad (67)$$

Solving the quadratic equation (67) for α gives

$$\alpha = \frac{(1-2a\lambda) \pm \sqrt{1-4a\lambda}}{2\lambda} \qquad (68)$$

$$u(x) = \frac{1 \pm \sqrt{1-4a\lambda}}{2\lambda} \qquad (69)$$

The following conclusions can be made here:
1. Using $\lambda = 0$ into (63) gives the exact solution $u(x) = 0$. However, $u(x)$ is undefined by using $\lambda = 0$ into (69). The point $\lambda = 0$ is called the singular point of Equation (63).
2. For $\lambda = \frac{1}{4a}$, Equation (69) gives only one solution $u(x) = 2a$. The point $\lambda = \frac{1}{4a}$ is called the bifurcation point of the equation. This shows that for $\lambda < \frac{1}{4a}$ then Equation (63) gives two real solutions but has no real solutions for $\lambda > \frac{1}{4a}$.
3. For $\lambda < \frac{1}{4a}$ Equation (69) gives two exact real solutions.

b. The Adomian Decomposition Method:
The Adomian decomposition method has been outlined before in previous chapters and has been applied to a wide class of linear Fredholm integral equations and linear Fredholm integro-differential equations. The method usually decomposes the unknown function $u(x)$ into an infinite sum of components that will be determined recursively through iterations as discussed before. The Adomian decomposition method will be applied in this chapter to handle nonlinear Fredholm integral equations.

Although the linear term $u(x)$ is represented by an infinite sum of components, the nonlinear terms such as $u^2, u^3, u^4 \sin u\, e^u$, sin *u*, etc. that appear in the equation, should be expressed by a special representation, called the Adomian polynomials $A_n, n \ge 0$. Adomian introduced a formal algorithm to establish a reliable representation for all forms of nonlinear terms. In what follows we present a brief outline for using the Adomian decomposition method for solving the nonlinear Fredholm integral equation

$$u(x) = f(x) + \lambda \int_a^b K(x,t)\, F\big(u(t)\big) dt, \qquad (70)$$

Where $F\big(u(t)\big)$ is a nonlinear function of $u(x)$ .The nonlinear Fredholm integral equation contains the linear term $u(x)$ and the nonlinear function. The linear term $u(x)$ can be represented normally by the decomposition series

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \qquad (71)$$

where the components $u_n(x), n \geq 0$ can be easily computed in a recursive manner as discussed before. However, the nonlinear term $F\big(u(x)\big)$ should be represented by the so-called Adomian polynomials A_n by using the algorithm

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^2}\left[F\left(\sum_{i=0}^{n} \lambda^i u_i\right)\right]\lambda = 0 \ , \quad n = 0,1,2,\dots \qquad (72)$$

So we obtained:

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_a^b K(x,t) \left(\sum_{n=0}^{\infty} A_n(t)\right) dt. \qquad (73)$$

To determine the components $u_0(x), u_1(x), \dots,$ we use the following recurrence relation

$$\begin{aligned} u_0(x) &= f(x), \\ u_1(x) &= \lambda \int_a^b K(x,t)\, A_0(t) dt, \\ u_2(x) &= \lambda \int_a^b K(x,t)\, A_1(t) dt, \qquad (73) \\ &\vdots \\ u_{n+1}(x) &= \lambda \int_a^b K(x,t)\, A_n(t) dt, \quad n \geq 0 \end{aligned}$$

Recall that in the modified decomposition method, a modified recurrence relation is usually used, where $u(x)$ is decomposed into two components $f_1(x)$ and $f_2(x)$, such that $f(x) = f_1(x) + f_2(x)$. In this case the modified recurrence relation becomes in the form

$$\begin{aligned} u_0(x) &= f_1(x), \\ u_1(x) &= f_2(x) + \lambda \int_a^b K(x,t)\, A_0(t) dt, \\ u_2(x) &= \lambda \int_a^b K(x,t)\, A_1(t) dt, \qquad (74) \\ &\vdots \\ u_{n+1}(x) &= \lambda \int_a^b K(x,t)\, A_n(t) dt, \quad n \geq 0. \end{aligned}$$

Having determined the components, the solution in a series form is readily obtained. The obtained series solution may converge to the exact solution if such a solution exists, otherwise the series can be used for numerical purposes. The following remarks can be observed:

(i) The convergence of the decomposition method has been examined in the literature by many authors.

(ii) The decomposition method always gives one solution, although the solution of the nonlinear Fredholm equation is not unique. The decomposition method does not address the existence and uniqueness concepts.

(iii)The modified decomposition method and the noise terms phenomenon can be used to accelerate the convergence of the solution. Generally speaking, the Adomian decomposition method is reliable and effective to handle differential and integral equations. This will be illustrated by using the following examples.

Example(5.2):

Use the Adomian decomposition method to solve the nonlinear Fredholm integral equation

$$u(x) = a + \lambda \int_0^1 u^2(t)\, dt, \qquad a > 0.$$

The Adomian polynomials for the nonlinear term u^2 are given by

$$\begin{aligned} A_0(x) &= {u_0}^2(x), \\ A_1(x) &= 2u_0(x)u_1(x), \end{aligned}$$

and so on. Substituting the series and the Adomian polynomials (15.91) into the left side and the right side of respectively we find

$$\sum_{n=0}^{\infty} u_n(x) = a + \lambda \int_0^1 \sum_{n=0}^{\infty} A_n(t)\, dt.$$

Using the Adomian decomposition method we set

$$u_0(x) = a, \ \ u_{k+1}(x) = \lambda \int_0^1 \sum_{n=0}^{\infty} A_k(t)\, dt. \quad k \geq 0$$

This in turn gives

$$u_0(x) = a,$$
$$u_1(x) = \lambda \int_0^1 u_0{}^2(t)\,dt = \lambda a^2,$$
$$u_2(x) = \lambda \int_0^1 (2u_0(t)u_1(t))\,dt = 2\lambda^2 a^3,$$
$$u_3(x) = \lambda \int_0^1 (2u_0(t)u_2(t) + u_1{}^2(t))\,dt = 5\lambda^3 a^4,$$
$$u_4(x) = \lambda \int_0^1 (2u_0(t)u_3(t) + 2u_1(t)u_2(t))\,dt = 14\lambda^4 a^5,$$
$$\vdots$$

The solution in a series form is given by

$$u(x) = a + \lambda a^2 + 2\lambda^2 a^3 + 5\lambda^3 a^4 + 14\lambda^4 a^5 + \cdots,$$
$$u(x) = \frac{1-\sqrt{1-4a\lambda}}{2\lambda}, \quad 0 < \lambda \le \frac{1}{4a}.$$

It is clear that only one solution was obtained by using the Adomian decomposition method. However, by using the direct computation method we obtained two solutions for this nonlinear problem as shown

6. Nonlinear Fredholm Integral Equations of the First Kind:

The standard form of the nonlinear Fredholm integral equations of the first kind is given by

$$f(x) = \int_a^b K(x,t)F(u(t))\,dt, \qquad (75)$$

where the kernel $k(x,t)$and the function $f(x)$ are given real-valued functions, and $F(u(x))$ is a nonlinear function of $u(x)$. The linear Fredholm integral equation of the first kind is presented where the homotopy perturbation method was used for handling this type of equations.

To determine a solution for the nonlinear Fredholm integral equation of the first kind we first convert it to a linear Fredholm integral equation of the first kind of the form

$$f(x) = \int_a^b K(x,t)v(t)\,dt, \quad x \in D \qquad (76)$$

by using the transformation

$$v(x) = F\big(u(x)\big). \qquad (77)$$

We assume that $F(u(x))$ is invertible, then we can set

$$u(x) = F^{-1}\big(v(x)\big). \qquad (78)$$

The linear Fredholm integral equation of the first kind has been investigated. An important remark has been reported in [3] and other references concerning the data function $f(x)$. The function $f(x)$. must lie in the range of the kernel $K(x,t)$ [3]. For example, if we set the kernel by

$$K(x,t) = e^x \sin t\ . \qquad (79)$$

Then if we substitute any integrable function $F(u(x))$ and we evaluate the integral, the resulting $f(x)$. must clearly be a multiple of e^x [3]. This means that if $f(x)$. is not a multiple of the x component of the kernel, then a solution does not exist. This necessary condition on $f(x)$. can be generalized. In other words, the data function $f(x)$. must contain components which are matched by the corresponding x components of the kernel $K(x,t)$.

Nonlinear Fredholm integral equation of the first kind is considered illposed problem because it does not satisfy the following three properties:

1. Existence of a solution.
2. Uniqueness of a solution.
3. Continuous dependence of the solution $u(x)$ on the data $f(x)$.

This property means that small errors in the data $f(x)$ should cause small errors [4] in the solution $u(x)$. The three properties were postulated by Hadamard [5]. Any problem that satisfies the three aforementioned properties is called well-posed problem. For any ill-posed problem, a very small change on the data $f(x)$ can give a large change in the solution $u(x)$ .This means that nonlinear Fredholm integral equation of the first kind may lead to a lot of difficulties.Several methods have been used to handle the linear and the nonlinear Fredholm integral equations of the first kind. The Legendre wavelets, the augmented Galerkin method, and the collocation method are examples of the methods used to handle this equation. The methods that we used so far in this text cannot handle this kind of equations independently if it is expressed in its standard form .However, in this text, we will first apply the method of regularization that received a considerable amount of interest, especially in solving first order integral equations. We will second apply the homotopy perturbation method [6] to handle specific cases of the Fredholm integral equations where the kernel $K(x,t)$ is separable.

In what follows we will present a brief summary of the method of regularization and the homotopy perturbation method that will be used to handle the Fredholm integral equations of the first kind.

a. The Method of Regularization:
The method of regularization was established independently by Phillips [7] and Tikhonov [8]. The method of regularization consists of replacing ill-posed problem by well-posed problem. The method of regularization transforms the linear Fredholm integral equation of the first kind

$$f(x) = \int_a^b K(x,t)\, v(t)dt, \qquad x \in D, \qquad (80)$$

to the approximation Fredholm integral equation

$$\mu v_\mu(x) = f(x) - \int_a^b K(x,t)\, v_\mu(t)dt,\ x \in D,\ (81)$$

Where μ is a small positive parameter. It is clear that (15.179) is a Fredholm integral equation of the second kind that can be rewritten

$$v_\mu(x) = \frac{1}{\mu} f(x) - \frac{1}{\mu}\int_a^b K(x,t)\, v_\mu(t)dt,\ x \in D,\ (82)$$

Moreover, it was proved in [1,3,9].
converges to the solution $v(x)$ of (15.178) as $\mu \to 0$ according to the following Lemma [10]:

Lemma (6.1):
Suppose that the integral operator is continuous and coercive in
the Hilbert space where $f(x)$, $u(x)$, and $v_\mu(x)$ are defined, then:
1. $|v_\mu|$is bounded independently of μ , and
2. $|v_\mu(x) - v(x)|$ when $\mu \to 0$.

The proof of this lemma can be found in [3, 9–10].
In summary, by combining the method of regularization with any of the methods used before for solving Fredholm integral equation of the second kind, we can solve Fredholm integral equation of the first kind.
The method of regularization transforms the first kind equation to a second kind equation. The resulting integral equation can be solved by any method that was presented before. The exact solution $v(x)$ of (15.178) can thus be obtained by

$$v(x) = \lim_{\mu\to 0} v_\mu(x). \qquad (83)$$

In what follows we will present four illustrative examples where we will use the method of regulariztion to transform the first kind integral equation to a second kind integral equation. The resulting equation will be solved by any appropriate method that we used before.

Example (6.2):
Combine the method of regularization and the direct computation method to solve the nonlinear Fredholm integral equation of the first kind

$$e^x = \int_0^{\frac{1}{2}} 2e^{x-4t} u^2(t)\, dt,$$

We first set

$$v(x) = u^2(x), u(x) = \pm\sqrt{v(x)}\,,$$

to carry out (15.182) into

$$e^x = \int_0^{\frac{1}{2}} 2e^{x-4t}\ v(t)dt.$$

Using the method of regularization, Equation (15.184) can be transformed to

$$v_\mu(x) = \frac{1}{r} e^x - \frac{1}{\mu}\int_0^{\frac{1}{2}} 2e^{x-4t}\ v_\mu(x)dt.$$

To use the direct computation method, The equation can be written as

$$v_\mu(x) = \left(\frac{1}{\mu} - \frac{\alpha}{\mu}\right) e^x,$$

Where

$$\alpha = \int_0^{\frac{1}{2}} 2e^{-4t}\ v_\mu(t)dt.$$

$$\alpha = \left(\frac{1}{\mu} - \frac{\alpha}{\mu}\right)\int_0^{\frac{1}{2}} 2e^{-3t}\ dt.$$

$$\alpha = \frac{2\left(1 - e^{\frac{1}{2}}\right)}{2-(2+\mu)e^{\frac{1}{2}}}$$

This in turn gives

$$v_\mu(x) = \frac{1}{\mu}\left(1 - \frac{2\left(1 - e^{\frac{1}{2}}\right)}{2-(2+\mu)e^{\frac{1}{2}}}\right)e^x.$$

The exact solution $v(x)$ can be obtained by

$$v(x) = \lim_{\mu\to 0} v_\mu(x) = \frac{e^{x+\frac{1}{2}}}{2\left(e^{\frac{1}{2}}-1\right)}.$$

Using (15.183) gives the exact solution by

$$u(x) = \pm\sqrt{\frac{e^{x+\frac{1}{2}}}{2\left(e^{\frac{1}{2}}-1\right)}}$$

Two more solutions to the equation are given by

$$u(x) = \pm e^{2x}.$$

b. The modified Adomian Decomposition Method:

The modified Adomian decomposition method [11–12] was frequently and thoroughly used in this text. The method decomposes the linear terms $u(x)\ v(x)$ and by an infinite sum of components of the form

$$u(x) = \sum_{n=0}^{\infty} u_n(x), v(x) = \sum_{n=0}^{\infty} v_n(x), \qquad (84)$$

where the components $u_n(x)$ and $v_n(x)$ will be determined recurrently. The method can be used in its standard form, or combined with the noise terms phenomenon. However, the nonlinear functions F_i and $\widetilde{F_i}$ for $i = 1,2,$ should be replaced by the Adomian polynomials A_n defined by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}[F(\sum_{i=0}^{n}\lambda^i u_i)]\lambda = 0, \quad n = 0,1,2,\ldots, \qquad (85)$$

Substituting the aforementioned assumptions for the linear and the nonlinear terms into the system and using the recurrence relations we can determine the components $u_n(x)$ and $v_n(x)$. Having determined these components, the series solutions and the exact solutions are readily obtained.

Example (6.3):

Use the modified Adomian decomposition method to solve the following system of nonlinear Fredholm integral equations

$$u(x) = \sin x - \pi + \int_0^{\pi}\big((1+xt)u^2(t) + (1-xt)v^2(t)\big)\,dt,$$

$$v(x) = \cos x + \frac{\pi^2}{2}x + \int_0^{x}\big((1-xt)u^2(t) + (1+xt)v^2(t)\big)\,dt.$$

Substituting the linear terms $u(x)$ and $v(x)$ and the nonlinear terms $u^2(x)$ and $v^2(x)$ respectively gives

$$\sum_{n=0}^{\infty} u_n(x) = \sin x - \pi + \int_0^{\pi}\big((1+xt)\sum_{n=0}^{\infty} A_n(t) + (1+xt)\sum_{n=0}^{\infty} B_n(t)\big)\,dt,$$

$$\sum_{n=0}^{\infty} v_n(x) = \cos x + \frac{\pi^2}{2}x + \int_0^{\pi}\big((1-xt)\sum_{n=0}^{\infty} A_n(t) - (1+xt)\sum_{n=0}^{\infty} B_n(t)\big)\,dt.$$

The modified decomposition method will be used here, hence we set the recursive relation

$$u_0(x) = \sin x, \qquad v_0(x) = \cos x,$$

$$u_1(x) = -\pi + \int_0^{\pi}((1+xt)\,u_0{}^2(t) \ + (1-xt)v_0{}^2(t)\ )dt = 0,$$

$$v_1(x) = \frac{\pi^2}{2}x + \int_0^{\pi}((1+xt)\,u_0{}^2(t) \ + (1-xt)v_0{}^2(t)\ )dt = 0.$$

This in turn gives the exact solutions

$$(u(x), v(x)) = (\sin x, \cos x).$$

Results:

The results that are obtained in our work show that the proper solution which gives the best approximation to solve nonlinear integral equations.

Conclusion:

The numerical methods for solving nonlinear integral equations characterized with an easy, fast and accuracy is high comparing with the manual solution. The results that are obtained in our work show that the numerical methods for solving nonlinear integral equations give the best approximation to solve the nonlinear integral equations.

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