CONSTRUCTION OF NEW Ω-SPECTRUM AND THEIR APPLICATIONS TO COHOMOLOGY THEORY

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ABSTRACT

The aim of this paper is to construct a new $\Omega -$ *spectrum associated with infinite symmetric product functor of connected CW-complexes and their application to cohomology theory.*
In this paper we studies all cohomology operation for the cohomology theory associated with new Ω*-spectrum.*
More precisely we prove that:

i) the abelian group of all cohomology operations of degree k for the cohomology theory $H^*(\;; \underline{A})$ *is isomorphic to the group* $H^{n+k}(SP^{\infty}(\Sigma^n Y); \underline{A})$ *; and*
ii) the graded abelian group of all stable cohomology operations of degree k for the cohomology theory $H^*(;\underline{A})$ *is isomorphic to the group* $\varprojlim H^{n+k}(SP^{\infty}(\Sigma^n Y); \underline{A})$.

KEYWORDS: Ω *- spectrum, Suspecsion spectrum, Sphere spectrum, Eilenberg-MacLane spectrum , Spectrul Homology theory,*

1. INTRODUCTION

In this paper we assume that **C** be the category whose objects are pointed topological spaces having the homotopy type of pointed finite CW-complexes and morphisms are maps of such spaces. Let $SP^{\infty} : \mathbf{C} \to \mathbf{C}$ be the infinite symmetric product functor; $\Sigma : \mathbf{C} \to \mathbf{C}$ be the suspension functor and **Ω** be the loop functor.

Dold and Tom [5] showed that $SP^{\infty}(S^n) = K(\mathbf{Z}, n)$, which is an **Eilenberg- MacLane** space with homotopy group $\pi_k(K(\mathbf{Z}, n))$ are all zero except for k = n.

P.K.Rana [7] showed that SP^n and SP^{∞} are covariant functor from the category of pointed topological spaces and base point preserving continuous maps to the category of pointed topological spaces and base point preserving continuous maps.

E.H. Spanier[8] showed that if X be a connected CW-complex the there exists a weak-homotopy equivalence from $SP^{\infty}(X)$ *to* $\Omega SP^{\infty}(\Sigma X)$.

Let E = $\{E_n, \alpha_n\}$ be a **Ω**-Spectrum and let X be connected CW-Complex, define a new **Ω**-spectrum $\underline{A} = \{A_k, \alpha_k\}$, where

$$A_k = \begin{cases} \Omega^{-(k-1)} SP^{\infty}(X), if\ k < 0 \\ \Omega SP^{\infty}(\Sigma^k X), \quad if\ k \geq 0 \end{cases}$$

The homotopy equivalence $\alpha_k : A_k \to \Omega A_{k+1}$ is defined by

$$\alpha_k = \begin{cases} identity, if\ k < 0 \\ \Omega \circ \rho_k, \quad k \geq 0 \end{cases}$$

, where $\rho_k : SP^{\infty}(\Sigma^k X) \to \Omega SP^{\infty}(\Sigma^{k+1} X)$ is an weak homotopy equivalence.

P.K.Rana and B. Mondal [9] showed that the generalized cohomology theory has some relations with the ordinary singular homology theory with integral coefficients.

In this paper we studies cohomology operations in the generalized cohomology theory.

We show that there exist relations between the cohomology operations and the general cohomology groups of some spaces in this general cohomology theory.

Now we recall the following definitions and statements:

Definition 1.1

Let X be a topological spaces with base point $x_0 \in X$. For $n \geq 0$, we define the n fold symmetric product of X , denoted by $SP^n X$ by $SP^0 X = x_0$, $SP^n X = X^n / S_n$ for $n \geq 1$, where X^n denotes the n fold cartesian product of X with itself and S_n denotes the symmetric group on n objects regarding as acting on X^n by permuting the coordinates.

Hence for $n \geq 1$, $SP^n X = \{(x_1, \cdots, x_n) : x_i \in X\}\}$,

We define $\lim_{n\to\infty} SP^n X = \bigcup_{n=1}^{\infty} SP^n X$ is called an infinite symmetric product of X and is denoted by $SP^{\infty} X$,

Definition 1.2

Let X be a topological spaces with base point $x_0 \in X$. Then the reduced suspension of X , denoted by ΣX is defined to be the quotient space $X \times I$ in which $(X \times \{1\}) \cup (\{x_0\} \times I) \cup (X \times \{0\})$ is identified to a single point.

Definition 1.3

Let X be a pointed topological space with base point x_0 . Then the Loop space of X denoted by ΩX, defined to the space of all continuous pointed map $\alpha : S^1 \to (X, x_0)$ equipped with compact open topology.

Definition 1.4

A spectrum E = $\{E_n, \alpha_n\}$ of spaces in C is said to be an $\Omega - spectrum$, if $\alpha_n : E_n \to \Omega E_{n+1}$, $n \in$ Z is a base point preserving weak homotopy equivalence for every integer n.

Definition 1.5

The spectrum E = $\{X_n, \alpha_n\}$ given by $X_n = K(Z, n)$ and $\alpha_n : K(Z, n) \to \Omega K(Z, n+1)$, a base point preserving weak homotopy equivalence, is called an **Eilenberg MacLane spectrum**.

Dold and Tom [5] showed that $SP^{\infty}(S^n) = K(\mathbf{Z}, n)$,which is an **Eilenberg- MacLane** space with homotopy group $\pi_k(K(\mathbf{Z}, n))$ are all zero except for k = n.

E.H. Spanier[8] showed that if X be a connected CW-complex then there exists a weak-homotopy equivalence from $SP^{\infty}(X) \to \Omega SP^{\infty}(\Sigma X)$

Definition 1.6

A cohomology operation in $H^*(\ ; \underline{A})$ of degree k is a natural transformation

$\varphi_m : H^m(\ ; \underline{A}) \to H^{m+k}(\ ; \underline{A})$.

Definition 1.7

For the cohomology theory $H^*(\ ; \underline{A})$, a stable cohomology operation of degree k is a sequence $\varphi_m : H^m(\ ; \underline{A}) \to H^{m+k}(\ ; \underline{A})$ of cohomology operation of degree k such that $\sigma^{m+k}(\varphi_m(x)) = \varphi_{m+1}(\sigma^m(x))$, for all $x \in H^m(X, A)$ and for all $X \in C$, where σ^m is the suspension isomorphism in $H^*(\ ; \underline{A})$.

Definition 1.8
A sequence of homomorphisms
$\mu_{m+k} : H^{m+k+1}(SP^{\infty}(\Sigma^{m+1}Y); \underline{A}) \longrightarrow H^{m+k}(SP^{\infty}(\Sigma^{m}Y); \underline{A})$ as the product of homomorphisms
$H^{m+k+1}(SP^{\infty}(\Sigma^{m+1}Y); \underline{A}) \xrightarrow{\overline{(\rho_m)}^*} H^{m+k+1}(SP^{\infty}(\Sigma^{m}Y); \underline{A}) \xrightarrow{(\sigma^{m+k})^{-1}} H^{m+k}(SP^{\infty}(\Sigma^{m}Y); \underline{A})$ i.e., $\mu_{m+k} = \overline{(\rho_m)}^* \circ (\sigma^{m+k})^{-1}$
is the homomorphism induced by the adjoint map $\overline{(\rho_m)}$ of $\rho_m : SP^{\infty}(\Sigma^{m}Y) \longrightarrow \Omega\, SP^{\infty}(\Sigma^{m}Y)$.
Hence the following sequence of abelian groups and homomorphisms
$\cdots \longrightarrow H^{m+k+1}(SP^{\infty}(\Sigma^{m+1}Y); \underline{A}) \xrightarrow{\pi_{m+k}} H^{m+k}(SP^{\infty}(\Sigma^{m}Y); \underline{A}) \rightarrow \cdots$ form an inverse system of groups and homomorphisms.

Let $\varprojlim H^{m+k}(SP^{\infty}(\Sigma^{m}Y); \underline{A})$ denote the inverse limit group of the above system.

Definition 1.9
The spectrum $E = \{X_n, \bar{\alpha}_n\}$ defined $X_n = \Sigma^n X$ and $\alpha_n : \Sigma(\Sigma^n X) \rightarrow \Sigma^{n+1}X$ to the natural homeomorphism is called a suspension spectrum.

Definition 1.10
If $X_n = S^n$ in the suspension spectrum E = $\{X_n, \alpha_n\}$, then the spectrum is called sphere spectrum and it is denoted by $\underline{S}$.
Thus the sphere spectrum $\underline{S} = \{S^n, \alpha_n\}$, where $\alpha_n : \Sigma S^n \longrightarrow S^{n+1}$ is the identity map.

Proposition 1.11
For any pointed topological space,we have $\pi_n(X) = \pi_{n-1}(\Omega X)$
Proof: Since

$$\begin{aligned}\pi_{n-1}(\Omega X) &= [S^{n-1}, \Omega X]\\ &= [\Sigma S^{n-1}, X]\\ &= [S^n, X]\\ &= \pi_n(X)\end{aligned}$$

From this we have the following:

Proposition 1.12
For any pointed topological space,we have $\pi_n(X) = \pi_{n-k}(\Omega^k X)$

Proposition 1.13
Let X be a connected CW-Complex and if $SP^{\infty}(X)$ is an Eilenberg-MacLane space of type K(G,n),then $SP^{\infty}(\Sigma X)$ is an Eilenberg-MacLane Space of type K(G,n+1).
Proof: Using [8],there is an weak homology equivalence,

$$\rho : SP^{\infty}(X) \longrightarrow \Omega SP^{\infty}(\Sigma X).$$

This implies that $\pi_m(SP^{\infty}(X)) = \pi_m(\Omega SP^{\infty}(\Sigma X)),\ \forall m \in Z$.
Thus using **Proposition 1.11** , we have $\pi_m(SP^{\infty}(X)) = \pi_{m+1}(SP^{\infty}(\Sigma X))$. Thus

$$\pi_{m+1}(SP^{\infty}(\Sigma X)) = \pi_m\ (SP^{\infty}(X) = \begin{cases} G, if\ m = n\\ 0, if\ \ m \neq n\end{cases}$$

Thus $SP^{\infty}(\Sigma X)$ is an Eilenberg-MacLane Space of type K(G, n+1).

From this proposition we conclude that
Remark:
Let X be a connected CW-Complex and if $SP^{\infty}(X)$ is an Eilenberg-MacLane space of type K(G,n),then $SP^{\infty}(\Sigma^k X)$ is also an Eilenberg-MacLane Space of type K(G,n+k).

2. Construction of New Ω-spectrum associated with infinite symmetric product functor .

Let E =$\{E_n, \alpha_n\}$ be a Ω-Spectrum and let X be connected CW-Complex, define a new Ω-spectrum $\underline{A} = \{A_k, \alpha_k\}$, where

$$A_k = \begin{cases} \Omega^{-(k-1)} SP^{\infty}(X), if\ k < 0\\ \Omega SP^{\infty}(\Sigma^k X), \qquad if\ k \geq 0\end{cases}$$

The homotopy equivalence $\alpha_k : A_k \longrightarrow \Omega A_{k+1}$ is defined by

$$\alpha_k = \begin{cases} identity, if\ k < 0\\ \Omega \circ \rho_k, \qquad k \geq 0\end{cases}$$

, where $\rho_k : SP^{\infty}(\Sigma^k X) \longrightarrow \Omega SP^{\infty}(\Sigma^{k+1} X)$ is an weak homotopy equivalence defined by,
$\rho_k(x_1, x_2, \cdot\cdot\cdot, x_k)(t) = ((x_1, t), (x_2, t), \cdot\cdot\cdot, (x_k, t)),\ \forall\ t \in I.$

The cohomology group $H^n(X; \underline{A})$ of $X \in C$ associated with this spectrum $\underline{A}$ is the direct limit of the sequence of groups and homomorphisms

$\cdots [X, A_n] \xrightarrow{\alpha_n} [X, \Omega A_{n+1}] \xrightarrow{(\Omega\circ\alpha_n)_*} [X, \Omega^2 A_{n+2}] \to \cdots$

Hence $H^n(X; \underline{A}) = \lim_{k\to\infty}[X, \Omega^k A_{n+k}] = [X, A_n]$, as each α_k is a weak homotopy equivalences.

Define the n-th reduced generalized cohomology group $H^n(X; \underline{A})$ associated with an $\Omega - spectrum$ by $H^n(X; \underline{A}) = [X, A_n]$

Let $(CO)^k{}_m$ be the set of all cohomology operations of degree k of the type φ_m for the cohomology theory $H^*(\,; \underline{A})$, and $(CO_s)^k$ be the set of all stable cohomology operations for the cohomology theory $H^*(\,; \underline{A})$.

Now we have:

3. Application to cohomology theory associated with New Ω −*spectrum* .

In this section we show that

i) In the ordinary singular cohomology theory $H^*(; Z)$, the groups $(CO)^k$ and $H^{m+k}(K(Z, m), Z)$ are isomorphic ,where $K(Z, m)$ is an Eilenberg- Maclane space.

ii) the groups $(CO)^k{}_m$ and $\varprojlim H^{n+k}(SP^\infty(\Sigma^n Y); \underline{A})$ are isomorphic.

Thus we have the following theorem:

Theorem 3.1

Let $(CO)^k{}_m$ be the set of all cohomology operations of degree k of the type φ_m for the cohomology theory $H^*(; \underline{A})$.Then $(CO)^k{}_m$ is a group.

Proof: Now we define '+' on $(CO)^k{}_m$ by the rule

$(\varphi_m + \phi_m)(X)(x) = (\varphi_m)(X)(x) + (\phi_m)(X)(x), \forall x \in H^m(X; \underline{A})$

and for all $X \in C$, where the right hand side addition is the addition in the additive abelian group $H^{m+k}(X, \underline{A})$. Then $(CO^k, +)$ is an additive group.

Let [Id] is the homotopy class of the identity map

$Id : SP^\infty(\Sigma^m X) \to SP^\infty(\Sigma^m X)$.

Now we have the following:

Theorem 3.2

The mapping $\lambda : (CO)^k \to H^{m+k}(SP^\infty(\Sigma^m X); \underline{A})$ be defined by $\lambda(\varphi) = \varphi[Id]$ is an isomorphism of groups.

Proof: Since [Id] is the homotopy class of the identity map

$Id : SP^\infty(\Sigma^m X) \to SP^\infty(\Sigma^m X)$ and let $x \in H^m(X, \underline{A})$ be represented by a map $g : X \to SP^\infty(\Sigma^m X)$ in C .

Now we define a map $\mu : H^{m+k}(SP^\infty(\Sigma^m X); \underline{A}) \to (CO)^k$ by $\mu(\alpha)(x) = [\alpha \circ g] = g^*(\alpha)$, for all $\alpha \in H^{m+k}(SP^\infty(\Sigma^m X)$

Now $\mu(\alpha + \beta)(x) = g^*(\alpha) + g^*(\beta) \Rightarrow \mu$ is a homomorphism. Again $\lambda \circ \mu$ = identity and $\mu \circ \lambda$ = identity and hence λ is an isomorohism and its inverse μ.

Using this theorem we have the following

Proposition 3.3

For the ordinary singular cohomology theory $H^*(\,; Z)$, the groups $(CO)^k$ and $H^{m+k}(K(Z, m), Z)$ are isomorphic ,where $K(Z, m)$ is an Eilenberg- Maclane space.

Proof: For X = S^0, the space $SP^\infty(\Sigma^m X)$ becomes the Eilenberg-Maclane space $K(Z, m)$ and the general cohomology theory reduces to the ordinary singular cohomology theory $H^*(\,; Z)$. Using the **Theorem 3.2,** it follows.

Proposition 3.4

Let $(CO_s)^k{}_m$ be the set of all stable cohomology operations for the cohomology theory $H^*(\,; A)$, then $(CO_s)^k{}_m$ forms an additive abelian group.

Proof: We define an addition '+' on $(CO_s)^k{}_m$ by the rule $(\varphi_m+\phi_m)(X)(x) = (\varphi_m)(X)(x) + (\phi_m)(X)(x)$, *for all* $x \in H^m(X; \underline{A})$ *and for all* $X \in C$.

Thus $(CO_s)^k{}_m$ becomes an additive abelian group.

Proposition 3.5

The graded abelian group $(CO_s)^k{}_m$ of all stable cohomology operations of degree k for the cohomology theory $H^*(\,; A)$ is isomorphic to the group $\varprojlim H^{n+k}(SP^\infty(\Sigma^n X); \underline{A})$.

Proof: Using the definition 1.8, it follows that an element of $\varprojlim H^{n+k}(SP^\infty(\Sigma^n X); \underline{A}); A)$ is a sequence

$x_m \in H^{n+k}(SP^\infty(\Sigma^n X); \underline{A})$ such that $\mu_{m+k}(x_{m+1}) = x_m$.

Hence $(\sigma^{m+k})^{-1} \circ (\rho_m)^*(x_{m+1}) = x_m$, *i.e.*, $(\rho_m)^*(x_{m+1}) = \sigma^{m+k}(x_m)$.

We now show that to each sequence of the elements $x_m \in H^{n+k}(SP^\infty(\Sigma^n X); \underline{A})$, there corresponds a stable cohomology operation of degree k in $(CO_s)^k{}_m$ and conversely.

Since $\mu : H^{m+k}(SP^\infty(\Sigma^m X); \underline{A}) \to (CO)^k{}_m$ be the homomorphism and so $\mu(x_m) = \varphi_m$. Now we show that $\{\varphi_m\}$ is a stable cohomology operation of degree k in $H^*(\,; \underline{A})$.Let $x \in H^*(\,; \underline{A})$ be represented by a map

$f : X \to SP^{\infty}(\Sigma^m X)$.Then $\sigma^m(x)$ is represented by the composite map

$\Sigma X \xrightarrow{\Sigma f} \Sigma SP^{\infty}(\Sigma^m X) \xrightarrow{\rho_m} SP^{\infty}(\Sigma^{m+1} X)$.Again $\varphi_m(x) = \mu(x_m)(x) = f^*(x_m)$. Hence we have $\varphi_{m+1}(\sigma^m(x)) = \mu(x_{m+1})(\sigma^m(x)) = (\rho_m \circ \Sigma f)^*(x_{m+1}) = (\Sigma f)^* \circ (\overline{\rho_m})^*(x_{m+1}) = (\Sigma f)^*(\sigma^{m+k}(x_m)) = \sigma^{m+k}(f^*(x_m)) = \sigma^{m+k}(\varphi_m(x)), \forall x \in H^m(X; \underline{A})$

$\Rightarrow \varphi_{m+1} \circ \sigma^m = \sigma^{m+k} \circ \varphi_m \Rightarrow \{\varphi_m\} \in \{(CO_s)^k_m\}$.

Conversely let $\{\varphi_m\} \in \{(CO_s)^k_m\}$, then $\varphi_{m+1}(\sigma^m(x)) = \sigma^{m+k}(\varphi_m(x)), \forall x \in H^m(X; \underline{A})$. Let $\lambda(\varphi_m) = x_m$,then $x_m \in H^{m+k}(SP^{\infty}(\Sigma^m X); \underline{A})$.

$\varphi_{m+1}(\sigma^m(x)) = \mu(x_{m+1})(\sigma^m(x)) = (\overline{\rho_m} \circ \Sigma f)^*(x_{m+1}) = (\Sigma f)^* \circ (\overline{\rho_m})^*(x_{m+1})$.

Again $\varphi_{m+1}(\sigma^m(x)) = \sigma^{m+k}(\mu(x_m)(x)) = \sigma^{m+k}(f^*(x)) = (\Sigma f)^* \circ \sigma^{m+k}(x_m)$.

Hence it follows that corresponding to each sequence φ_m, there exists a sequence of elements $x_m \in H^{m+k}(SP^{\infty}(\Sigma^m X); \underline{A})$ such that $\sigma^{m+k}(x_m) = (\rho_m)^*(x_{m+1})$.

Proposition 3.6

If $X = S^n$, then the graded abelian group $\{(CO_s)^k{}_m\}$ of all stable cohomology operations of degree k in $H(;A)$ is isomorphic to lim $H^{m+k+n}(K(Z, m+n); Z)$,where $H^*(; Z)$ is the ordinary cohomology theory with coefficient in Z.

Proof: Using the **theorem 3.5** and since $H^{m+k}(SP^{\infty}(\Sigma^m S^n); \underline{A}) = H^{m+k}(K(Z, m+n); \underline{A}) = H^{m+k+n}(K(Z, m+n); Z)$, it follows.

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REFERENCES

[1] Adhikari,A.and Rana,P.K.(2001);A study of functors associated with Topological groups, Studia Univ. "Babes-Bolyai," Mathematica, Vol XLVI, No 4, Dec, 2001.
[2] Adhikari,M.R.;Construction of general homology theory, Cal.Math.Society 71(6)1979,396-398;
[3] Adhikari,M.R.;Basic Algebraic Topology and its Applications, Springer,2016.
[4] Albrecht Dold ;Homology of Symmetric Products and Other Functors of Complexes,Vol.68,No.1 (Jul.1958),54-80;
[5] Dold and Thom;.... Ann of Math 67(1958),239-281;
[6] Hatcher,Allen,Algebraic Topology,Cambridge University Press,2002;
[7] Rana, P.K.(2012):-A study of some functors and their relations,Journal of Indian Acad. Math.Vol.34,No.1(2012) pp.73-81;
[8] Spanier,E.H.;Infinite symmetric product of functors spaces and dual- ity,Ann.of Math 69(1959),142-198;
[9] Rana,Pravanjan Kumar and Mandal Bhaskar:-Homology theory as- sociated with new Ω-spectrum ,Journal of the Calcutta Mathematical Society,17,(1)1-8,2021