Independent Domination in Some Operations on Bipolar Fuzzy Graphs

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Abstract

In this paper, the bounds of an independent domination number in some operations on bipolar fuzzy graphs like join, Cartesian product, composition, cross product and strong product were obtained.

Keywords: Bipolar fuzzy graph independent domination number in operations on bipolar fuzzy graphs like join, Cartesian product, strong product and composition.
Classification 2010: 03E72, 68R10, 68R05

1 Introduction

In(1994) Zhang [13,14] initiated the concept of bipolar fuzzy sets as a generalization of fuzzy sets. Bipolar fuzzy sets are an extension of fuzzy sets whose membership degree range is [-1, 1]. In a bipolar fuzzy set, the membership degree 0 of an element means that the element is irrelevant to the corresponding property, the membership degree (0, 1] of an element indicates that the element somewhat satisfies the property and the membership degree [-1, 0) of an element indicates that the element somewhat satisfies the implicit counter-property. Akram [1,2,3,4] introduced and studied the notations of bipolar fuzzy graph, bipolar fuzzy graphs with applications, regular bipolar fuzzy graph and metric in bipolar fuzzy graphs. A. Somasundaram and S. Somasundaram [11] introduced and discussed the concept of domination in fuzzy graphs. The independent domination number and irredundance number in graphs are introduced by Cockayne [6] and Hedetniemi [7]. Nagoorgani and Vadivel [9] introduced and discussed the concepts of domination, independent domination and irredundance in fuzzy graphs using strong edges. The concept of domination in Intuitionistic fuzzy graphs was investigated by Parvathi and Thamizhendhi [10]. The concepts of domination, independence and irredundance number in bipolar fuzzy graph by Akarm and at al (2013)[5]. In (2020) Mansour and Mahioub Shubatah [8] initiated the concepts of independent dominating and chromatic number in bipolar fuzzy graph and investigated the relationship between this concept and the others in bipolar fuzzy graphs.

The aim of this paper is to introduce the concept of independent domination number in some operation of bipolar fuzzy graphs.Such us join, Cartesian product, strong product and composition.

2 Preliminaries

In this section, we review some basic definitions and terminology related to bipolar fuzzy graphs and independent domination in bipolar fuzzy graph.

Definition 2.1:[1] A bipolar fuzzy graph (BFG) is of the form $G = (V, E)$ where (i) $V = \{v_1, v_2, ..., v_n\}$ such that $\mu_1^+ : X \rightarrow [0, 1]$ and $\mu_1^- : X \rightarrow [-1, 0]$ (ii)$E \subset V \times V$ where $\mu_2^+ : V \times V \longrightarrow [0, 1]$ and $\mu_2^- : V \times V \longrightarrow [-1, 0]$ such that

$\mu_{2ij}^+ = \mu_2^+(v_i, v_j) \leq \min(\mu_1^+(v_i), \mu_1^+(v_j))$ and $\mu_{2ij}^- = \mu_2^-(v_i, v_j) \geq \max (\mu_1^-(v_i), \mu_1^-(v_j))$, $\forall (v_i, v_j) \in$ E

Definition 2.2:[1] A bipolar fuzzy graph $G = (V, E)$ is called strong if

$\mu_2^+(v_i, v_j) = \min(\mu_1^-(v_i), \mu_1^-(v_j))$

and $\mu_2^-(v_i, v_j) = \max(\mu_1^+(v_i), \mu_1^+(v_j))$, $\forall (v_i, v_j) \in E$

Definition 2.3:[8] Let G be a BFG and $u, v \in V(G)$, Then u, v are said to be adjacent if there is strong edge between them.

Definition 2.4:[5] An edge (V, E) is said to be strong edge in BFG, $G = (V, E)$ if

$\mu_2^+(u, v) \geq (\mu_2^+)^\infty(u, v)$ and $\mu_2^-(u, v) \leq (\mu_2^-)^\infty(u, v)$ where

$(\mu_2^+)^\infty(u, v) = max(\mu_2^+)^k(u, v) : k = 1, 2, ..., n$ and

$(\mu_2^-)^\infty(u, v) = min(\mu_2^-)^k(u, v) : k = 1, 2, ..., n.$

Definition 2.5:[5] Let $G = (V, E)$, be a BFG on V. And $u, v \in V$, we say that u dominates v in G if there exists a strong edge between them.

Definition 2.6:[5] A subset S of $V(G)$ is called a dominating set of bipolar fuzzy graph G if for every $v \in V - S$, there exists $u \in S$ such that u dominates v.

Definition 2.7:[5] A dominating set S of a BFG, $G = (V, E)$ is said to be minimal dominating set if $S - v$ is not dominating set $\forall v \in S$.

Definition 2.8:[5] Two vertices u and v in a BFG, $G = (V, E)$, are said to be independent if there is no strong edge between them.

Definition 2.9:[5] Minimum cardinality among all minimal dominating set is called domination number of G, and is denoted by $\gamma(G)$.

Note: A minimal dominating set D of a bipolar fuzzy graph G with $|D| = \gamma(G)$ is the minimum dominating set of G and is denoted by $\gamma(G)-$set of G.

Definition 2.10:[5]A subset S of V in bipolar fuzzy graph G is said to be an independent set if

$(\mu_2^+)(u,v) < (\mu_2^+)^\infty(u,v)$ and $(\mu_2^-)(u,v) > (\mu_2^-)^\infty(u,v) \forall u,v \in S$

Definition 2.11:[5]An independent set S of BFG, $G(V,E)$ is said to be maximal independent, if for every vertex $v \in V - S$, the set $S \cup \{v\}$ is not independent.

Definition 2.12:[5] Let $G = (V,E)$ be a bipolar fuzzy graph. Then the cardinality of G is defined to be $|G| = \sum_{v_i \in V} \frac{1 + \mu_1^+(v_i) + \mu_1^+(v_i)}{2} + \sum_{(v_i,v_j) \in E} \frac{1 + \mu_2^+(v_i,v_j) + \mu_2^+(v_i,v_j)}{2}$.

Definition 2.13:[8] The Order of bipolar fuzzy graph is denoted by P and is defined as $P(G) = \sum_{i=1}^{n} (\frac{1+\mu_1^+(v_i)+\mu_1^-(v_i)}{2})$, n is number of vertices in G and the size of G is denoted by q and is defined as

$q = |E| = \sum_{(v_i,v_j) \in E} \frac{1 + \mu_2^+(v_i,v_j) + \mu_2^+(v_i,v_j)}{2}$.

Definition 2.14:[8] A dominating set D in BFG, $G = (V,E)$ is said to be an independent dominating set if D is an independent.

Definition 2.15:[8] An independent dominating set D of a BFG $G = (V,E)$ is called minimal independent dominating set if $D - \{u\}$ is not dominating $\forall u \in D$

Definition 2.16:[8] The minimum fuzzy cardinality taken ever all independent dominating set in bipolar fuzzy graph G is called the independence domination number of G and is denoted by $\gamma_i(G)$.

Note: A minimal independent dominating set D of a bipolar fuzzy graph G with $|D| = \gamma_i(G)$ is called the minimum independent dominating set of G and is denoted by $\gamma_i(G)-$set of G.

Definition 2.17:[8] In a BFG G, a vertex u and edge e are said to be incident if u is the end vertex of e and if they are incident, then they are said to cover each other.

Definition 2.18:[8] Let $G = (V, E)$ be any BFG. A vertex subset s of V which covers all edges in G is called a vertex cover of G. A vertex cover set S of bipolar fuzzy graph G is called minimal cover vertex set if $S - \{v\}$ is not cover vertex set $\forall v \in S$.

The minimum fuzzy cardinality among all minimal vertex cover sets in bipolar fuzzy graph G is called the vertex covering number and is denoted by $\alpha_0(G)$.

3 independent dominating in Some Operation on bipolar fuzzy graphs

In this section we introduce and study the concept of independent dominating in Some Operation on bipolar fuzzy graphs such as the Join, the Cartesian product, the Composition, and strong product.

Definition 3.1:[12] Let $A_1 = (\mu^+_{A_1}, \mu^-_{A_1})$ and $A_2 = (\mu^+_{A_2}, \mu^-_{A_2})$ be two bipolar fuzzy subsets of V_1 and V_2 in which $V_1 \cap V_2 = \phi$ and let $B_1 = (\mu^+_{B_1}, \mu^-_{B_1})$ and $B_2 = (\mu^-_{B_2}, \mu^-_{B_2})$ be two bipolar fuzzy subsets of $V_1 \times V_2$ and $V_2 \times V_1$ respectivily, then, we denoted the join of two bipolar fuzzy graphs G_1 and G_2 by $G_1 + G_2 = (A_1 + A_2, B_1 + B_2)$ and defined as follows

$$\begin{cases} (\mu^+_{A_1} + \mu^+_{A_2})(x) = min\{\mu^+_{A_1}(x), \mu^+_{A_2}(x)\} \\ (\mu^-_{A_1} + \mu^-_{A_2})(x) = max\{\mu^-_{A_1}(x), \mu^-_{A_2}(x)\} \end{cases} \text{if } x \in V_1 \cup V_2$$

$$\begin{cases} (\mu^+_{B_1} + \mu^+_{B_2})(xy) = min\{\mu^+_{B_1}(xy), \mu^+_{B_2}(xy)\} \\ (\mu^-_{B_1} + \mu^-_{B_2})(xy) = max\{\mu^-_{B_1}(xy), \mu^-_{B_2}(xy)\} \end{cases} \text{if } xy \in E_1 \cap E_2.$$

$$\begin{cases} (\mu^+_{B_1} + \mu^+_{B_2})(xy) = min\{\mu^+_{B_1}(xy), \mu^+_{B_2}(xy)\} \\ (\mu^-_{B_1} + \mu^-_{B_2})(xy) = max\{\mu^-_{B_1}(xy), \mu^-_{B_2}(xy)\} \end{cases} \text{if } xy \in E'.$$

Where E' is the set of all edges joining the vertx of V_1 and V_2.

Theorem 3.2: Let G_1 and G_2 be two bipolar fuzzy graphs and D_1 be γ_i-set of G_1, D_2 be γ_i-set of G_2.

Then $\gamma_i(G_1 + G_2) = min(\gamma_i(G_1), \gamma_i(G_2))$

Proof: Let D_1 be a γ_i-set of G_1, and D_2 be γ_i-set of G_2.

$(\mu^+_{B_1} + \mu^+_{B_2})(uv) = min(\mu^+_{B_1}(u), \mu^+_{B_2}(v))$

$(\mu^-_{B_1} + \mu^-_{B_2})(uv) = max(\mu^-_{B_1}(u), \mu^-_{B_2}(v))$

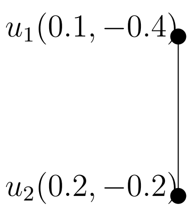
Since D_1 is an independent dominating set of G_1.

Then D_1 is an independent dominating set of $G_1 + G_2$.

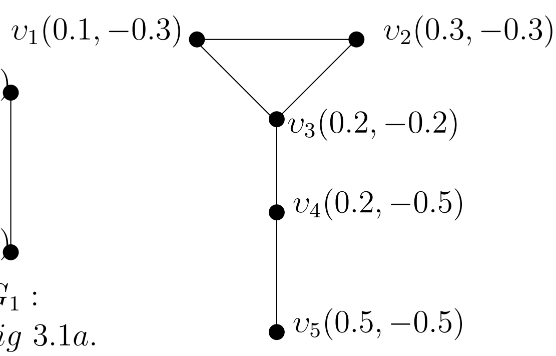
Similarly D_2 is an independent dominating set of $G_1 + G_2$.

Hence $\gamma_i(G_1 + G_2) = min(\gamma_i(G_1), \gamma_i(G_2))$

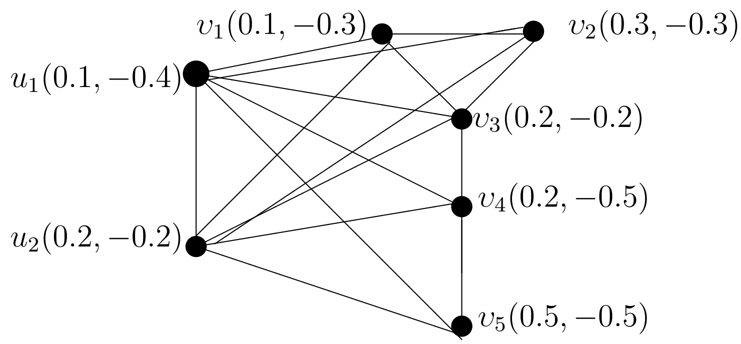
Exampe 3.1: Consider a bipolar fuzzy graphs G_1, G_2 and $G_1 + G_2$ given in figures 3.1a, 3.1b, and 3.1c respectively such that all edges in G_1 and G_2 are effective.



G_1 :
$Fig\ 3.1a.$



G_2 :
$Fig\ 3.1b.$



$G_1 + G_2$

$Fig\ 3.1c.$

In figures 3.1a and 3.1b we see that $D_1 = \{u_1\}$ is γ_i- set of G_1 and $D_2 = \{v_1, v_2\}$ is γ_i- set of G_2.Hence $\gamma_i(G_1 + G_2) = \{u_1\}$.

Definition 3.3:[12] The Cartesian product $G_1 \times G_2$ of two bipolar fuzzy graphs G_1 and G_2 is a bipolar fuzzy graph $G = (A, B)$ of a pair of bipolar fuzzy sets defined on the Cartesian product $G_1 \times G_2$ such that

$\mu_A^+(x_1, x_2) = min(\mu_{A_1}^+(x_1), \mu_{A_2}^+(x_2))$

$\mu_A^-(x_1, x_2) = max(\mu_{A_1}^-(x_1), \mu_{A_2}^-(x_2)) \quad \forall (x_1, x_2) \in V_1 \times V_2;$

and

$\mu_B^+(x, x_2)(x, y_2) = min(\mu_{B_1}^+(x), \mu_{B_2}^+(x_2, y_2))$

$\mu_B^-(x, x_2)(x, y_2) = max(\mu_{B_1}^-(x), \mu_{B_2}^-(x_2, y_2)) \quad \forall x_2y_2 \in E_2;$

$\mu_B^+(x_1, z)(y_2, z) = min(\mu_{B_1}^+(x_1y_1), \mu_{A_2}^+(z))$

$\mu_B^-(x_1, z)(y_2, z) = max(\mu_{B_1}^-(x_1y_1), \mu_{A_2}^-(z))$ for all $x_1y_1 \in E_1$

Theorem 3.4: Let D_1 and D_2 be the minimum independent dominating set of bipolar fuzzy graphs and G_1 and G_2 respectively.

Then $\gamma_i(G_1 \times G_2) = |D_1 \times D_2| + |S_1 \times S_2| \quad : |S_1| = \beta_0(G_1)$

. $\quad |S_2| = \beta_0(G_2)$

Proof: Let D_1 be a γ_i-set of G_1 and D_2 be a γ_i-set of G_2.

Then $D_1 \times D_2$ is an independent and it is not dominate $G_1 \times G_2$. So there exist a vertex subset of $(V_1 \times V_2)$ say $S_1 \times S_2$ such that

$|S_1| = \beta_0(G_1)$, $|S_2| = \beta_0(G_2)$ and $(D_1 \times D_2) \cup (S_1 \times S_2)$ is an independent and dominating set of $G_1 \times G_2$.

Hence $\gamma_i(G_1 \times G_2) = |D_1 \times D_2| + |S_1 \times S_2|$

Examples 3.2: Consider the two bipolar fuzzy graphs G_1, G_2given in figures 3.2a, 3.2b respectively.

And the figures 3.2c give the cartesian product of G_1 and G_2

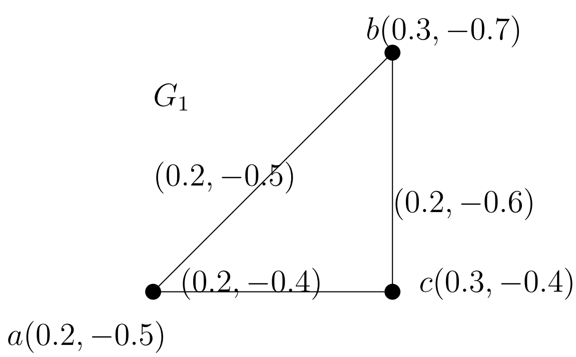


Fig 3.2*a*

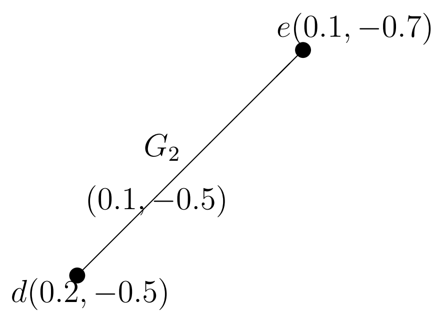
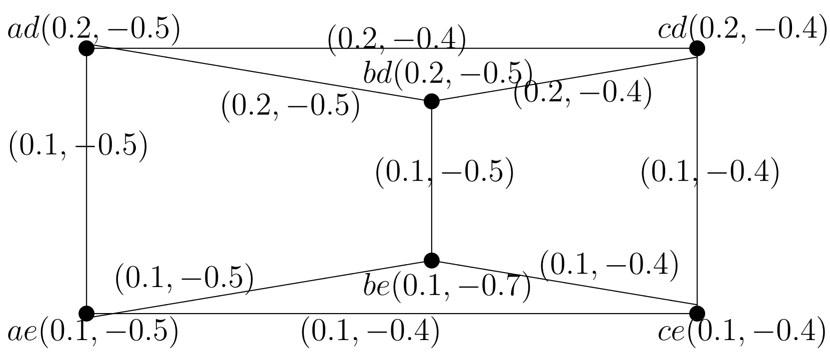


Fig 3.2*b*.



$G_1 \times G_2$

Fig 3.2*c*.

We see that $D_1 = \{a\}$ is γ_i- set of G_1,
$D_2 = \{e\}$ is γ_i- set of G_2.
$D_1 \times D_2$ it is not dominates $G_1 \times G_2$ but $(D_1 \times D_2) \cup (S_1 \times S_2)$ is a γ_i- set of $G_1 \times G_2 = \{ae, cd\}$.

Definition 3.5:[12]The composition $G_1[G_2]$ is the pair (A, B) of bipolar fuzzy sets defined on the composition $G_1[G_2]$ such that

$$\mu_A^+(x_1, x_2) = min(\mu_{A_1}^+(x_1), \mu_{A_2}^+(x_2))$$
$$\mu_A^-(x_1, x_2) = max(\mu_{A_1}^-(x_1), \mu_{A_2}^-(x_2)) \quad \forall (x_1, x_2) \in V_1 \times V_2$$

$$\mu_B^+(x, x_2)(x, y_2) = min(\mu_{B_1}^+(x), \mu_{B_2}^+(x_2, y_2))$$
$$\mu_B^-(x, x_2)(x, y_2) = max(\mu_{B_1}^-(x), \mu_{B_2}^-(x_2, y_2)) \quad \forall x_2y_2 \in E_2$$

$$\mu_B^+(x_1, z)(y_2, z) = min(\mu_{B_1}^+(x_1y_1), \mu_{A_2}^+(z))$$
$$\mu_B^-(x_1, z)(y_2, z) = max(\mu_{B_1}^-(x_1y_1), \mu_{A_2}^-(z))$$

$$\mu_B^+((x_1, x_2)(y_1, y_2)) = min(\mu_{A_2}^+(x_2), \mu_{A_2}^+(y_2), \mu_{B_1}^+(x_1y_1))$$
$$\mu_B^-((x_1, x_2)(y_1, y_2)) = min(\mu_{A_2}^-(x_2), \mu_{A_2}^-(y_2), \mu_{B_1}^-(x_1y_1))$$

In the following theorem we gives γ_i of the composition of two bipolar fuzzy graphs

Theorem 3.6: Let G_1 and G_2 be a bipolar fuzzy graphs and let D_1 be γ_i- set of G_1 and D_2 be γ_i- set of G_2
then $\gamma_i(G_1 o G_2) = |D_1 \times D_2|$.

Proof: Let $(a, b) \notin D_1 \times D_2$
Case(i): $a \notin D_1$ and $b \in D_2$
Let $a \notin D_1$, there exist $a_1 \in D_1$ such that a_1 dominates a.
Then

$\mu^+_{B_1}(a,a_1) = min(\mu^+_{A_1}(a),\mu^+_{A_1}(a_1))$

$\mu^-_{B_1}(a,a_1) = max(\mu^-_{A_1}(a),\mu^-_{A_1}(a_1))$

Now $(a_1,b) \in D_1 \times D_2$,

$\mu^+_B((a,b)(a_1,b)) = \mu^+_{B_1}(a,a_1) \wedge \mu^+_{A_2}(b)$

$$= min((\mu^+_{A_1}(a),\mu^+_{A_1}(a_1)) \wedge \mu^+_{A_2}(b))$$

$$= \mu^+_{A_1}(a) \wedge \mu^+_{A_2}(b) \wedge \mu^+_{A_1}(a_1) \wedge \mu^+_{A_2}(b)$$

$$= (\mu_{A_1} o \mu_{A_2})^+(a,b) \wedge \mu_{A_1} o \mu_{A_2})^+(a_1,b)$$

$\mu^-_B((a,b)(a_1,b)) = \mu^-_{B_1}(a,a_1) \vee \mu_{A_2}(b)$

$$= max((\mu^-_{A_1}(a),\mu^-_{A_1}(a_1)) \vee \mu^-_{A_2}(b))$$

$$= \mu^-_{A_1}(a) \vee \mu^-_{A_2}(b) \vee \mu_{A_1}(a_1) \vee \mu^-_{A_2}(b)$$

$$= (\mu_{A_1} o \mu_{A_2})^-(a,b) \vee \mu_{A_1} o \mu_{A_2})^-(a_1,b)$$

Hence (a_1,b) dominates (a,b)

Case(ii): $a \in D_1$ and $b \notin D_2$

Let $b_2 \in D_2$, such that

$\mu^+_{B_2}(b,b_2) = min(\mu^+_{A_2}(b),\mu^+_{A_2}(b_2))$

$\mu^-_{B_2}(b,b_2) = max(\mu^-_{A_2}(b),\mu^-_{A_2}(b_2))$

Now $(a,b_2) \in D_1 \times D_2$,

$\mu^+_B((a,b)(a,b_2)) = min(\mu^+_{A_1}(a),\mu^+_{B_2}(b,b_2))$

$$= min((\mu^+_{A_1}(a),\mu^+_{A_2}(b)) \wedge \mu^+_{A_2}(b_2))$$

$$= \mu^+_{A_1}(a) \wedge \mu^+_{A_2}(b) \wedge \mu^+_{A_1}(a) \wedge \mu^+_{A_2}(b_2)$$

$$= (\mu_{A_1} o \mu_{A_2})^+(a,b) \wedge (\mu_{A_1} o \mu_{A_2})^+(a,b_2)$$

$\mu^-_B((a,b)(a,b_2)) = max(\mu^-_{A_1}(a),\mu^-_{B_2}(b,b_2))$

$$= \mu^-_{A_1}(a) \vee \mu^-_{A_2}(b) \vee \mu^-_{A_2}(b_2)$$

. $\mu_B^-((a,b)(a,b_2)) = \mu_{A_1}^-(a) \vee \mu_{A_2}^-(b) \vee \mu_{A_1}^-(a) \vee \mu_{A_2}^-(b_2)$

$$= (\mu_{A_1} o \mu_{A_2})^-(a,b) \vee (\mu_{A_1} o \mu_{A_2})^-(a,b_2)$$

Hence (a,b_1) dominates (a,b) Case(iii): $a \notin D_1$ and $b \notin D_2$

D_1 and D_2 be the minimum independent dominating sets of G_1 and G_2, respectively.

Therefore, there exist $a_1 \in D_1$ and $b_2 \in D_2$ such that

$\mu_{B_1}^+(a,a_1) = min(\mu_{A_1}^+(a), \mu_{A_1}^+(a_1))$

$\mu_{B_1}^-(a,a_1) = max(\mu_{A_1}^-(a), \mu_{A_1}^-(a_1))$

and $\mu_{B_2}^+(b,b_2) = min(\mu_{A_2}^+(b), \mu_{A_2}^+(b_2))$

$\mu_{B_2}^-(b,b_2) = max(\mu_{A_2}^-(b), \mu_{A_2}^-(b_2))$

Lat $(a_1,b_2) \in D_1 \times D_2$,

$\mu_B^+((a,b)(a_1,b_2)) = min(\mu_{A_2}^+(a), \mu_{A_2}^+(b_2), \mu_{B_1}^+(a,a_1))$

$$= min((\mu_{A_2}^+(b), \mu_{A_2}^+(b_2), \mu_{A_1}^+(a) \wedge \mu_{A_1}^+(a_1))$$

$$= \mu_{A_1}^+(a) \wedge \mu_{A_2}^+(b) \wedge \mu_{A_1}^+(a) \wedge \mu_{A_2}^+(b_2)$$

$$= (\mu_{A_1} o \mu_{A_2})^+(a,b) \wedge (\mu_{A_1} o \mu_{A_2})^+(a_1,b_2)$$

$\mu_B^-((a,b)(a_1,b_2)) = max(\mu_{A_2}^-(a), \mu_{A_2}^-(b_2), \mu_{B_1}^-(a,a_1))$

$$= max((\mu_{A_2}^-(b), \mu_{A_2}^-(b_2), \mu_{A_1}^-(a) \vee \mu_{A_1}^-(a_1))$$

$$= \mu_{A_1}^-(a) \vee \mu_{A_2}^-(b) \vee \mu_{A_1}^-(a) \vee \mu_{A_2}^-(b_2)$$

$$= (\mu_{A_1} o \mu_{A_2})^-(a,b) \vee (\mu_{A_1} o \mu_{A_2})^-(a_1,b_2)$$

Therefore (a_1,b_2) dominates (a,b) in $G_1 o G_2$. This implies that $D_1 \times D_2$ is an independent dominating set of $G_1 o G_2$.

Now we prove $D_1 \times D_2$ is minimum. Let $(z_1, z_2) \in D_1 \times D_2$, $z_1 \in D_1$ and $z_2 \in D_2$. By our assumption D_1 and D_2 are minimal independent dominating set of D_1 and D_2 respectively.

Therefore $D_1 - z_1$ and $D_2 - z_2$ are not an independent dominating set. Clearly we get $(D_1 \times D_2) - (z_1 \cup z_2)$ is not a minimal independent dominating set. This implies that $(D_1 \times D_2)$ is minimal independent dominating set of $G_1 o G_2$.

Therefore $\gamma_i(G_1 o G_2) = |D_1 \times D_2|$.

Examples 3.3: Consider the bipolar fuzzy graphs G_1, G_2 and G_3 given in figures 3.3a, 3.3b, and 3.3c respectively.

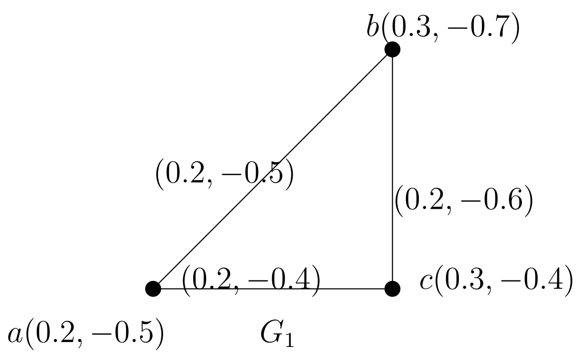


Fig 3.3*a*

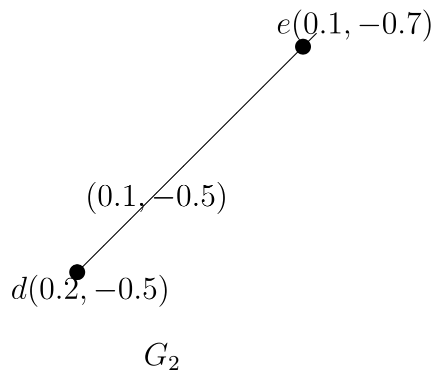


Fig 3.3*b*.

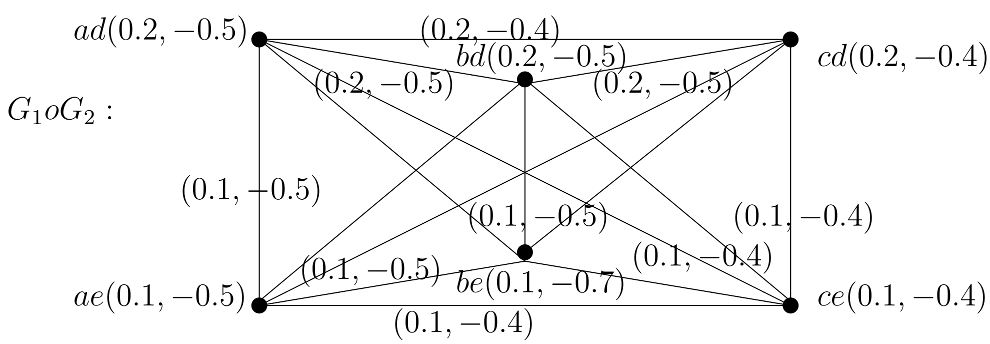


Fig 3.3*c*.

We see that

$D_1 = \{a\}$ is a γ_i-set of G_1

$D_2 = \{e\}$ is a γ_i-set of G_2

$D = \{ae\}$ is a γ_i-set of $G_1 o G_2$

Definition 3.7:[12] The strong product $G_1 \otimes G_2$ is the pair (A, B) of bipolar fuzzy sets defined on The strong product $G_1 \otimes G_2$ such that

$\mu_A^+(x_1,x_2) = min(\mu_{A_1}^+(x_1), \mu_{A_2}^+(x_2))$

$\mu_A^-(x_1,x_2) = max(\mu_{A_1}^-(x_1), \mu_{A_2}^-(x_2)) \quad \forall (x_1,x_2) \in V_1 \times V_2;$

$\mu_B^+(x,x_2)(x,y_2) = min(\mu_{B_1}^+(x), \mu_{B_2}^+(x_2,y_2))$

$\mu_B^-(x,x_2)(x,y_2) = max(\mu_{B_1}^-(x), \mu_{B_2}^-(x_2,y_2)) \quad \forall x_2y_2 \in E_2;$

$\mu_B^+(x_1,z)(y_2,z) = min(\mu_{B_1}^+(x_1y_1), \mu_{A_2}^+(z))$

$\mu_B^-(x_1,z)(y_2,z) = max(\mu_{B_1}^-(x_1y_1), \mu_{A_2}^-(z));$

$\mu_B^+((x_1,x_2)(y_1,y_2)) = min(\mu_{A_2}^+(x_2), \mu_{A_2}^+(y_2), \mu_{B_1}^+(x_1y_1))$

$\mu_B^-((x_1,x_2)(y_1,y_2)) = min(\mu_{A_2}^-(x_2), \mu_{A_2}^-(y_2), \mu_{B_1}^-(x_1y_1)).$

Theorem 3.8: Let G_1 and G_2 be a bipolar fuzzy graphs and D_1 is a γ_i-set of G_1 and D_2 is a γ_i-set of G_2

then $\gamma_i(G_1 \otimes G_2) = |D_1 \times D_2|$.

Proof: Let $(a,b) \notin D_1 \times D_2$

Case(i): $a \notin D_1$ and $b \in D_2$

If $a \notin D_1$, there exist $a_1 \in D_1$ such that a_1 dominates a.

Then,

$\mu_{B_1}^+(a,a_1) = min(\mu_{A_1}^+(a), \mu_{A_1}^+(a_1))$

$\mu_{B_1}^-(a,a_1) = max(\mu_{A_1}^-(a), \mu_{A_1}^-(a_1)).$

Now $(a_1,b) \in D_1 \times D_2$,

$\mu_B^+((a,b)(a_1,b)) = \mu_{B_1}^+(a,a_1) \wedge \mu_{A_2}^+(b)$

$$= min((\mu_{A_1}^+(a), \mu_{A_1}^+(a_1)) \wedge \mu_{A_2}^+(b))$$

$$= \mu_{A_1}^+(a) \wedge \mu_{A_2}^+(b) \wedge \mu_{A_1}^+(a_1) \wedge \mu_{A_2}^+(b)$$

$$= (\mu_{A_1} \otimes \mu_{A_2})^+(a,b) \wedge \mu_{A_1} \otimes \mu_{A_2})^+(a_1,b);$$

$\mu_B^-((a,b)(a_1,b)) = \mu_{B_1}^-(a,a_1) \vee \mu_{A_2}(b)$

$$= max((\mu^-_{A_1}(a), \mu^-_{A_1}(a_1)) \vee \mu^-_{A_2}(b))$$

$$= \mu^-_{A_1}(a) \vee \mu^-_{A_2}(b) \vee \mu_{A_1}(a_1) \vee \mu^-_{A_2}(b)$$

$$= (\mu_{A_1} \otimes \mu_{A_2})^-(a, b) \vee \mu_{A_1} \otimes \mu_{A_2})^-(a_1, b).$$

Hence (a_1, b) dominates (a, b)

Case(ii): $a \in D_1$ and $b \notin D_2$

If $b_2 \in D_2$, such that $\mu^+_{B_2}(b, b_2) = min(\mu^+_{A_2}(b), \mu^+_{A_2}(b_2))$

$\mu^-_{B_2}(b, b_2) = max(\mu^-_{A_2}(b), \mu^-_{A_2}(b_2))$.

Now $(a, b_2) \in D_1 \times D_2$,

$\mu^+_B((a, b)(a, b_2)) = min(\mu^+_{A_1}(a), \mu^+_{B_2}(b, b_2))$

$$= min((\mu^+_{A_1}(a), \mu^+_{A_2}(b)) \wedge \mu^+_{A_2}(b_2))$$

$$= \mu^+_{A_1}(a) \wedge \mu^+_{A_2}(b) \wedge \mu^+_{A_1}(a) \wedge \mu^+_{A_2}(b_2)$$

$$= (\mu_{A_1} \otimes \mu_{A_2})^+(a, b) \wedge (\mu_{A_1} \otimes \mu_{A_2})^+(a, b_2);$$

$\mu^-_B((a, b)(a, b_2)) = max(\mu^-_{A_1}(a), \mu^-_{B_2}(b, b_2))$

$$= \mu^-_{A_1}(a) \vee \mu^-_{A_2}(b) \vee \mu^-_{A_2}(b_2)$$

. $\mu^-_B((a, b)(a, b_2)) = \mu^-_{A_1}(a) \vee \mu^-_{A_2}(b) \vee \mu^-_{A_1}(a) \vee \mu^-_{A_2}(b_2)$

$$= (\mu_{A_1} \otimes \mu_{A_2})^-(a, b) \vee (\mu_{A_1} \otimes \mu_{A_2})^-(a, b_2).$$

Hence (a, b_1) dominates (a, b)

Case(iii): $a \notin D_1$ and $b \notin D_2$

D_1 and D_2 be the minimum independent dominating sets of G_1 and G_2 .

Therefore, there exist $a_1 \in D_1$ and $b_2 \in D_2$ such that

$\mu^+_{B_1}(a, a_1) = min(\mu^+_{A_1}(a), \mu^+_{A_1}(a_1))$
$\mu^-_{B_1}(a, a_1) = max(\mu^-_{A_1}(a), \mu^-_{A_1}(a_1))$
and $\mu^+_{B_2}(b, b_2) = min(\mu^+_{A_2}(b), \mu^+_{A_2}(b_2))$.
$\mu^-_{B_2}(b, b_2) = max(\mu^-_{A_2}(b), \mu^-_{A_2}(b_2))$ Lat $(a_1, b_2) \in D_1 \times D_2$,
$\mu^+_B((a, b)(a_1, b_2)) = min(\mu^+_{A_2}(a), \mu^+_{A_2}(b_2), \mu^+_{B_1}(a, a_1))$

$$= min((\mu^+_{A_2}(b), \mu^+_{A_2}(b_2), \mu^+_{A_1}(a) \wedge \mu^+_{A_1}(a_1))$$

$$= \mu^+_{A_1}(a) \wedge \mu^+_{A_2}(b) \wedge \mu^+_{A_1}(a) \wedge \mu^+_{A_2}(b_2)$$

$$= (\mu_{A_1} \otimes \mu_{A_2})^+(a, b) \wedge (\mu_{A_1} \otimes \mu_{A_2})^+(a_1, b_2);$$

$\mu^-_B((a, b)(a_1, b_2)) = max(\mu^-_{A_2}(a), \mu^-_{A_2}(b_2), \mu^-_{B_1}(a, a_1))$

$$= max((\mu^-_{A_2}(b), \mu^-_{A_2}(b_2), \mu^-_{A_1}(a) \vee \mu^-_{A_1}(a_1))$$

$$= \mu^-_{A_1}(a) \vee \mu^-_{A_2}(b) \vee \mu^-_{A_1}(a) \vee \mu^-_{A_2}(b_2)$$

$$= (\mu_{A_1} \otimes \mu_{A_2})^-(a, b) \vee (\mu_{A_1} \otimes \mu_{A_2})^-(a_1, b_2).$$

Therefore (a_1, b_2) dominates (a, b) in $G_1 \otimes G_2$. This implies that $D_1 \times D_2$ is an independent dominating set of $G_1 \otimes G_2$.
Now we have to prove $D_1 \times D_2$ is minimal. Let $(z_1, z_2) \in D_1 \times D_2$, $z_1 \in D_1$ and $z_2 \in D_2$. By our assumption D_1 and D_2 are minimal independent dominating set of D_1 and D_2, respectively.
Therefore $D_1 - \{z_1\}$ and $D_2 - \{z_2\}$ are not an independent dominating set. Clearly $(D_1 \times D_2) - \{(z_1, z_2)\}$ is not an independent dominating set of$G_1 \otimes G_2$. This implies that $(D_1 \times D_2)$ is minimal independent dominating set of $G_1 \otimes G_2$. Hence $\gamma_i(G_1 \otimes G_2) = |D_1 \times D_2|$.

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