DIFFERENTIAL EQUATION WITH POCHHAMMER POLYNOMIALS $(x)_n$ SOLUTIONS $(x)_1$ TO $(x)_4$

Vicente Aboites[1] and David Trejo-Garcia[2]

[1, 2]Center for Research in Optics
Loma del Bosque 115
Col. Campestre
León, Gto., 37150, MEXICO

Abstract: In this paper we present a third order differential equation that has the Pochhammer polynomials from orders one through four as solutions. Obtaining this equation is useful to characterize the Pochhammer polynomials in a similar way to other families of special polynomials. We believe that an equation for higher orders is possible following the same methodology.

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1. Introduction

The Pochhammer polynomials arise from the Pochhammer symbol, which is largely used in combinatorics. It is also present in definitions of the gamma and hypergeometric functions and the binomial coefficient. Pochhammer polynomials were studied in 1730 by Stirling and later by Appell. Their name recognizes Leo August Pochhammer who introduced the now conventional notation $(x)_n$. Alternative names are shifted factorial functions, rising factorials and upper factorials [1]. These polynomials are defined for all real x and all nonnegative integer n values, although a generalization to negative n is possible. These polynomials are defined by the n-fold product:

$$(x)_n = x(x+1)(x+2)\dots(x+n-1) = \prod_{j=0}^{n-1}(x+j). \tag{1}$$

An equivalent definition expresses $(x)_n$ in terms of a factorial function and a binomial coefficient:

$$(x)_n = n!\binom{x+n-1}{n}. \tag{2}$$

A generating function for the Pochhammer polynomials is:

$$\frac{1}{(1-t)^v} = \sum_{n=0}^{\infty} (v)_n \frac{t^n}{n!}. \tag{3}$$

It is also generated by repeatedly differentiating a power of which $-v$ is the exponent.

$$x^v \frac{d^n}{dx^n} x^{-v} = (v)_n \left(-\frac{1}{x}\right)^n. \tag{4}$$

The first five Pochhammer polynomials are:

$$(x)_0 = 1 \tag{5.1}$$
$$(x)_1 = x \tag{5.2}$$
$$(x)_2 = x^2 + x \tag{5.3}$$
$$(x)_3 = x^3 + 3x^2 + 2x \tag{5.4}$$
$$(x)_4 = x^4 + 6x^3 + 11x^2 + 6x \tag{5.5}$$

Complementary to its n-fold product definition, it would be interesting to find a differential equation able to give as solutions the Pochhammer polynomials, this in a similar way as other well-known differential equations and its polynomial solutions such as Hermite, Laguerre and Legendre, among others [2].

In this article a differential equation is obtained such that it has as solutions the Pochhammer polynomials $(x)_i$ with $i = 1$ to 4.

2. Methodology

We start by assuming that there exists a second order linear differential equation that has as solutions the polynomials $(x)_1$, $(x)_2$ and $(x)_3$. Such an equation should have the form:

$$(a_1x^2 + a_2x + a_3)y''(x) + (a_4x + a_5)y'(x) + ny(x) = 0 \tag{6}$$

We include the term $ny(x)$ to achieve similarity to other differential equations associated with special polynomials. Substituting the values of $(x)_1$, $(x)_2$ and $(x)_3$ in $y(x)$ and its derivatives, gives in each case a system of linear equations with $n + 1$ equations and 5 unknown values. On each case, some values are assumed to be zero in order to have a single solution. The most important solutions are shown in Table 1. For simplicity, the dependence of x in $y(x)$ is omitted from this point on.

Differential equation	Solution
$-xy' + y = 0$	$(x)_1 = x$
$\frac{1}{4}y'' - \frac{1}{2}(1+2x)y' + 2y = 0$	$(x)_2 = x^2 + x$
$\frac{1}{3}y'' - (1+x)y' + 3y = 0$	$(x)_3 = x^3 + 3x^2 + 2x$

Table 1: All polynomials can be solutions to several differential equations but these were chosen because they can be easily related.

These solutions are related to the next second order differential equation.

$$\frac{1}{4}\frac{(n-1)^2}{2n-3}y'' - \left[x + \frac{1}{2}(n-1)\right]y' + ny = 0 \tag{7}$$

We extend our focus to the case of a second order differential equation that has $(x)_4$ as a solution. If we apply the same methodology as before, we are left with a system of 5 linear equations for 5 unknown values, which implies a single solution. The resulting equation is as follows:

$$\frac{1}{25}(9 - 3x - x^2)y'' - \frac{1}{25}(33 + 22x)y' + 4y = 0. \tag{8}$$

It is not straightforward to associate this solution to equation (7), so we opt instead for a third order differential equation with the following structure:

$$(a_1x^3 + a_2x^2 + a_3x + a_4)y''' + (a_5x^2 + a_6x + a_7)y'' + (a_8x + a_9)y' + ny = 0. \tag{9}$$

By substituting the value of $(x)_4$ and its derivatives, we are left with a system of 5 linear equations with 9 unknown values. By conveniently adjusting the values of some of them, we obtain the following equation:

$$\left(-\frac{1}{60}x + \frac{1}{40}\right)y''' + \frac{9}{20}y'' - \left[x + \frac{3}{2}\right]y' + ny = 0. \tag{10}$$

We then associate equation (10) with (7) in order to have an equation that has as solutions the Pochhammer polynomials of orders one to four:

$$-\frac{n-3}{20}\left(\frac{1}{3}x + \frac{1}{2}\right)y''' + \frac{1}{4}\frac{(n-1)^2}{2n-3}y'' - \left[x + \frac{1}{2}(n-1)\right]y' + ny = 0 \tag{11}$$

Equations (7) and (11) are the main result of this paper: *i.e.* low order differential equations whose solutions are the Pochhammer polynomials $(x)_i$ with $i = 1$ to 4.

3. Conclusions

Through a clear methodology and using basic algebra, we were able to find both a differential equation of second order that has the Pochhammer polynomials of orders one to three as its solutions and a differential equation of third order that has the polynomials of orders one to four as its solutions. This set of differential equations could prove to be a straightforward way to deal with Pochhammer polynomials. They also help to further characterize these functions in a similar way to other sets of special polynomials.

Following the same methodology, it should be possible to obtain new sets of differential equations that have more orders of Pochhammer polynomials as solutions.

References

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