One Radius Theorem

Harmonic Function Theory

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Import
The present paper deals with the study of harmonic and analytical functions. It deals with well-known and powerful theorems of the Complex Analysis and has as its central theme the One-Radius Theorem, somehow reversing the Mean Value theorem of harmonic functions. These considerations are set out in *Mark A. Pinsky's* article [*Mean Values and the Maximum Principle: A Proof in Search of More Theorems*].

Purpose
Our goal is to prove that in the One-Radius Theorem, the precondition of continuity of u in closed $\overline{D}(R)$ and not simply in $D(R)$, is necessary.

Methodology
For this reason, we will give an example in which we construct a function u continuous to $D(R)$, which satisfies the other conditions of the theorem and yet it is not harmonic to $D(R)$.

Before proceeding with the presentation, we should recall basic concepts of Complex Analysis. We will formulate definitions and theorems that are simply referred or used in this paper.

Definition 0.0.1 *A set $S \subset \mathbb{C}$ is coherent if there are no subsets of $\mathbb{C}, A, B \neq \varnothing$, open to S with the following properties: $S = A \cup B$ and $A \cap B = \varnothing$. So, we call S a coherent set of $\mathbb{C}$ if this can not be written as a union of two foreign, non-empty and open to the S sets. Otherwise, S is called non-coherent. An open and coherent set is called a place.*

Definition 0.0.2 *A real function $u(x, y)$ that is twice productive and satisfies the Laplace equation*

$$u_{xx} + u_{yy} = 0$$

in a place D is called harmonic in D.

Definition 0.0.3 *We say that a function $u = u(x, y)$ from place Ω to $\mathbb{R}$ has the property of Mean Value in Ω, if it is true*

$$u(a, b) = \frac{1}{2\pi} \int_0^{2\pi} u(a + R\cos\theta, a + R\sin\theta)d\theta, \forall \overline{D}(a, b; R) \subset \Omega.$$

That is, the value of u in the center of the disk is equal to the average of its values on the periphery of the disk.

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Theorem 0.0.4 (Mean Value for Harmonic Functions) *If u is a harmonic function on disk $D(a, b; R) \subset \Omega$ and continuous in $\overline{\Omega}$, then it has the property of mean value in D.*

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One-Radius Theorems

We know that the Mean Value attribute characterizes the harmonic functions. In particular, if u u is a continuous function from the place Ω to $\mathbb{R}$ and satisfies the property of Mean Value in Ω , then it is harmonic and C^∞ in it. Then follows the One-Radius Theorem, a form of inversion of the Mean Value Theorem.

Theorem 0.0.5 (Theorem of One Radius) *Suppose $u = u(x, y)$) is a continuous function in the disk $\overline{D}(R)$. If $\forall (x, y) \in D$, $\exists\ r = r(x, y)$ with $0 < r \leq R - \sqrt{x^2 + y^2}$ and such that it is valid*

$$u(x, y) = \frac{1}{2\pi} \int_0^{2\pi} u(x + r\cos\theta, y + r\sin\theta) d\theta,$$

then u is harmonic and C^2 in $D(R)$.

Then we will show that in the theorem of one radius, the condition of continuity of u in the closed $\overline{D}(R)$ and not just in $D(R)$, is necessary. Indeed, below, we will give an example in which a continuous function u in $D(R)$ satisfies the other conditions of the theorem and is nonetheless harmonic in $D(R)$.

Example 0.0.6 *Suppose the function $u(z) = \log |z|$. u is continuous and harmonic at $\mathbb{C}^*$. Therefore, if*

$$\Delta = \{z \in \mathbb{C} : r_1 \leq |z| \leq r_2\}$$

is a ring, we can find a harmonic function in Δ that takes any fixed values in the inner and outer circle, putting $u(z) = b \log |z| + a$ for appropriate a, b. So we suppose the rings

$$\Delta_n = \{z \in \mathbb{C} : 1 - \frac{1}{2^n} \leq |z| \leq 1 - \frac{1}{2^{n+1}}\}, n = 0, 1, 2, ...$$

and define the function

$$u(z) = u_n(z), \forall z \in \Delta_n$$

with

$$u_n(z) = a_n + b_n \log |z| \,, n = 1, 2, ...$$

and

$$u_0(z) = \begin{cases} 0, 0 \le |z| \le \frac{1}{4} \\ a_0 + b_0 \log|z|, \frac{1}{4} \le |z| \le \frac{1}{2} \end{cases}$$

Where

$$a_0 = -\log\frac{1}{4}$$

$$b_n = n+1, n = 0, 1, 2, ...$$

$$a_n = a_0 - \sum_{m=0}^{n-1} c_m$$

$$c_m = \log(1 - \frac{1}{2^{m+1}}), m = 0, 1, 2, ...$$

We will show that u fulfills all the conditions of the theorem of one radius except one, the one of continuity in closed $\overline{D}(0,1)$. *In particular, we will show the following*

1) *u is continuous at* $D(0,1)$

2) *u is not continuous at* $\overline{D}(0,1)$

3) $\forall z \in D(0,1)$ *there is a radius* $r(z) > 0$ *such that u satisfies the property of mean value in disk* $D(z, r(z))$

4) *finally, and while all of the above applies, we will show that u is not harmonic to* $D(0,1)$.

Indeed,

1) $u_0(z)$, *as defined is continuous at* Δ_0. *In the inner of* Δ_n, $u(z) = u_n(z)$ *so u is continuous in the inner of each ring with the above properties. On the borders of the rings we have* $u_n(z) = u_{n+1}(z)$. *Actually if*

$$z \in \Delta_n \cap \Delta_{n+1}$$

then

$$|z| = 1 - \frac{1}{2^{n+1}}$$

And

$$u_{n+1}(z) = a_{n+1} + b_{n+1}\log(1 - \frac{1}{2^{n+1}}) = a_0 - \sum_{m=0}^{n} c_m + (n+2)\log(1 - \frac{1}{2^{n+1}})$$

$$= a_0 - \sum_{m=0}^{n-1} c_m - c_n + (n+1)\log(1 - \frac{1}{2^{n+1}}) + \log(1 - \frac{1}{2^{n+1}})$$

$$= a_0 - \sum_{m=0}^{n-1} c_m - \log(1 - \frac{1}{2^{n+1}}) + (n+1)\log(1 - \frac{1}{2^{n+1}}) + \log(1 - \frac{1}{2^{n+1}})$$

$$= a_0 - \sum_{m=0}^{n-1} c_m + (n+1)\log(1 - \frac{1}{2^{n+1}}) = a_n + b_n \log|z| = u_n(z).$$

Suppose z_0 *belongs in one of the borders of* Δ_n *for example at the right edge of* Δ_n *with* $|z_0| = 1 - \frac{1}{2^{n+1}}$. *Then* $z_0 \in \Delta_n \cap \Delta_{n+1}$. *Suppose* $\epsilon > 0$. u_n *is continuous at* z_0. *So there exists* $\delta_1 > 0$ *such as* $\forall z \in \Delta_n$ *with* $|z - z_0| < \delta_1 \Rightarrow |u_n(z) - u_n(z_0)| < \epsilon$. *Likewise* u_{n+1} *is continuous at* z_0. *So there is* $\delta_2 > 0$ *such as* $\forall z \in \Delta_{n+1}$ *with* $|z - z_0| < \delta_2 \Rightarrow |u_{n+1}(z) - u_{n+1}(z_0)| < \epsilon$. *We set* $\delta = \min\{\delta_1, \delta_2\}$ *and we have* $\delta \leq \delta_1, \delta_2$. *So there is* $\delta > 0$ *such as* $\forall z \in \Delta_n \cup \Delta_{n+1}$ *with* $|z - z_0| < \delta \Rightarrow |u(z) - u(z_0)| < \epsilon$. *Therefore* u *is continuous on the boundaries of the rings too and finally continuous at* $D(0,1)$.

2) *To show that* u *is not continuous at* $\overline{D}(0,1)$, *it is enough to show that there is no limit*

$$\lim_{|z|\to 1} u(z).$$

We suppose the sequence of points of $D(0,1)$

$$z_n = 1 - \frac{1}{2^n}$$

with

$$\lim_{n\to+\infty} z_n = 1.$$

We have

$$\lim_{n\to+\infty} u(z_n) = \lim_{n\to+\infty} u_n(z_n) = \lim_{n\to+\infty} (a_n + b_n \log|z_n|)$$

$$= \lim_{n\to+\infty} [a_0 - \sum_{m=0}^{n-1} c_m] + \lim_{n\to+\infty} [(n+1)\log(1 - \frac{1}{2^{n+1}})]$$

$$= a_0 - \lim_{n\to+\infty} \sum_{m=0}^{n-1} c_m + \lim_{n\to+\infty} n\log(1 - \frac{1}{2^{n+1}}) + \lim_{n\to+\infty} \log(1 - \frac{1}{2^{n+1}})$$

$$= a_0 + \sum_{m=0}^{+\infty} [-c_m] + \lim_{n\to+\infty} \log[(1 - \frac{1}{2^{n+1}})^n].$$

Now we will show that

$$\lim_{n\to+\infty} \log[(1 - \frac{1}{2^{n+1}})^n] = \frac{1}{e}$$

and

$$\sum_{m=0}^{+\infty} [-c_m] = +\infty.$$

For the fisrt we have

$$\forall n \in \mathbb{N}, 2^n \geq n \Rightarrow (1 - \frac{1}{2^n})^n \geq (1 - \frac{1}{n})^n$$

and therefore

$$\lim_{n\to+\infty} (1 - \frac{1}{2^n})^n \geq \lim_{n\to+\infty} (1 - \frac{1}{n})^n = \frac{1}{e}.$$

Otherwise

$$\forall n \in \mathbb{N}, 2^n \geq n \Rightarrow (1 - \frac{1}{2^n})^{2^n} \geq (1 - \frac{1}{2^n})^n$$

So

$$\frac{1}{e} = \lim_{n \to +\infty} (1 - \frac{1}{2^n})^{2^n} \geq \lim_{n \to +\infty} (1 - \frac{1}{2^n})^n$$

eventually by the combination of the above

$$\lim_{n \to +\infty} (1 - \frac{1}{2^n})^n = \frac{1}{e}$$

which is the first demand. Now we set

$$t_n = -c_n = -\log(1 - \frac{1}{2^{n+1}}), s_n = \frac{1}{n+1}$$

where

$$\forall n \in \mathbb{N}, 1 - \frac{1}{2^{n+1}} \leq 1 \Rightarrow t_n = -\log(1 - \frac{1}{2^{n+1}}) \geq 0.$$

We have

$$\lim_{n \to +\infty} [\frac{t_n}{s_n}] = \lim_{n \to +\infty} [\frac{-\log(1 - \frac{1}{2^{n+1}})}{\frac{1}{n+1}}]$$

$$= \lim_{n \to +\infty} [\log(1 - \frac{1}{2^{n+1}})^{-(n+1)}]$$

$$= \log[\frac{1}{\lim_{n \to +\infty}(1 - \frac{1}{2^{n+1}})^{(n+1)}}] = \log e = 1.$$

Thus according to Limit Comparison Test the series

$$\sum_{n=0}^{+\infty} t_n, \sum_{n=0}^{+\infty} s_n$$

have the same behavior. But

$$\sum_{n=0}^{+\infty} s_n = \sum_{n=0}^{+\infty} \frac{1}{n+1} = +\infty$$

finally

$$\sum_{n=0}^{+\infty} t_n = +\infty$$

which is the second demand.
Returning to the limit calculation, we have

$$\lim_{n \to +\infty} u(z_n) = a_0 + \infty + \frac{1}{e} = +\infty$$

therefore the limit

$$\lim_{|z|\to 1} u(z).$$

does not exist. 3)*In the inner of the rings, u is harmonic. Consequently the property of Mean value is satisfied. If* z_0 *is a point on the boundary of a ring, suppose* $z_0 \in \Delta_n \cap \Delta_{n+1}$*, then select a radius less than the width of the ring* $\Delta_{n+1}, r < \frac{1}{2^{n+2}}$ *(the width of the sequence of rings* Δ_n *decreases) the disk* $D(z_0, r)$ *is entirely in the inner of* $\Delta_n \cup \Delta_{n+1}$*. At* $D(0,1)$ *define the continuous function*

$$M(z) = \frac{1}{2\pi}\int_0^{2\pi} u(z + re^{i\theta})d\theta.$$

Let $\{z_k\}$ *be a sequence with elements of* Δ_n*with*

$$\lim_{k\to+\infty} z_k = z_0.$$

For all $k \in \mathbb{N}$ *we have ,* $M(z_k) \le M(z_0)$*. Also* u_n *is harmonic in* Δ_n *therefore* $\forall z_k \in \Delta_n, \exists r_k > 0$ *such that it is true*

$$u_n(z_k) = \frac{1}{2\pi}\int_0^{2\pi} u_n(z_k + r_k e^{i\theta})d\theta \le \frac{1}{2\pi}\int_0^{2\pi} u_n(z_k + re^{i\theta})d\theta = M(z_k) \le M(z_0)$$

Consequently

$$u_n(z_k) \le M(z_0), \forall k \in \mathbb{N} \Rightarrow \lim_{k\to+\infty} u_n(z_k) \le M(z_0)$$

and finally

$$u_n(z_0) \le M(z_0).$$

Then select $\{z_k\}$ *a sequence with elements of* Δ_{n+1} *with*

$$\lim_{k\to+\infty} z_k = z_0$$

like we did above, considering that now it is true $\forall k \in \mathbb{N}, M(z_k) \ge M(z_0)$ *we will get*

$$u_{n+1}(z_0) \ge M(z_0)$$

eventually

$$M(z_0) \le u_{n+1}(z_0) = u_n(z_0) \le M(z_0)$$

namely

$$u_n(z_0) = M(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta.$$

Therefore, the property of Mean Value is satisfied on the boundaries of the rings and as a result throughout $D(0,1)$*.*

4) *The last thing we have to prove is that u is not harmonic in* $D(0,1)$ *and we will do that by showing that*

$$\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}$$

don'At exist for the points on the border of the rings. Let $z = x + iy \in \Delta_n$. *Then*

$$u(z) = u_n(z) = a_n + b_n \log \sqrt{x^2 + y^2}$$

$$\frac{\partial u(z)}{\partial x} = \frac{x}{x^2 + y^2} b_n$$

Consider the border of Δ_k, $|z| = 1 - \frac{1}{2^k} = r_k$. *We suppose the sequences of points of* $D(0,1)$

$$z_n = r_k - \frac{1}{2^n}, w_n = r_k + \frac{1}{2^n}.$$

z_n *approaches this boundary of* Δ_k *through the ring* Δ_n, *while* w_n *through the ring* Δ_{n+1}. *So we have*

$$\lim_{n \to +\infty} \frac{\partial u(z_n)}{\partial x} = \frac{r_k}{r_k^2} b_k = \frac{b_k}{r_k} = \frac{k+1}{r_k}$$

$$\lim_{n \to +\infty} \frac{\partial u(w_n)}{\partial x} = \frac{r_k}{r_k^2} b_{k+1} = \frac{b_{k+1}}{r_k} = \frac{k+2}{r_k}$$

namely, along two different paths z_n, w_n *with* $|z_n| \longrightarrow r_k, |w_n| \longrightarrow r_k$ *we have*

$$\lim_{n \to +\infty} \frac{\partial u(z_n)}{\partial x} \neq \lim_{n \to +\infty} \frac{\partial u(w_n)}{\partial x}$$

Therefore the limit

$$\lim_{|z| \to r_k} \frac{\partial u(z)}{\partial x}$$

does not exist, which is sufficient to complete the proof.

Conclusions

The Mean Value property in bounded places , such as a disk , characterizes harmonic functions with basic precondition , to ensure the continuity in the closeness of this place.

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