The $\exp(-\varphi(z))$-expansion method for (3+1)-dimensional generalized Boiti-Leon-Manna-Pempinelli equation

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Abstract In this paper, we apply the $\exp(-\varphi(z))$-expansion method to obtain exact solutions of the (3+1)-dimensional generalized Boiti-Leon-Manna-Pempinelli equation, and then we give some computer simulations to illustrate our main results. Four types of exact solutions are obtained, which are hyperbolic, exponential, trigonometric and rational function solutions.

1 Introduction

The (3+1)-dimensional generalized Boiti-Leon-Manna-Pempinelli (gBLMP) equation [1] is given by:

$$u_{xxxy} - 3(u_x u_y)_x + 3u_{ys} - 2u_{yt} = 0. \tag{1}$$

The gBLMP equation is a meaningful nonlinear model for the incompressible fluid. If we take $s = t$, Eq.(1) can be reduced to BLMP equation [2] as follows:

$$u_{xxxy} - 3u_{xx}u_y - 3u_x u_{xy} + u_{yt} = 0. \tag{2}$$

Searching for exact solutions of nonlinear differential equation plays a pivotal part in studying nonlinear physical phenomena. In the past few years, there has been extraordinary progress in seeking exact solutions of NLDEs, such as the sine-cosine method [3], the bifurcation method of dynamic systems [4], the modified simple equation method [5], the enhanced $(\frac{G'}{G})$-expansion method [6], the complex method [7–10], the $\exp(-\varphi(z))$-expansion method [11, 12] and so on. In this article, we employ the $\exp(-\varphi(z))$-expansion method to derive exact solutions of the gBLMP equation, and then we give some computer simulations to illustrate our main results.

2 The $\exp(-\varphi(z))$-expansion method

Consider a nonlinear PDE as follows:

$$F(u, u_x, u_y, u_t, u_{xx}, u_{yy}, u_{tt}, \cdots) = 0, \tag{3}$$

where F is a polynomial of an unknown function $u(x, y, t)$ and its derivatives, and it contains highest order derivatives and nonlinear terms are involved.

Step 1. Substitute traveling wave transform

$$u(x, y, t) = w(z), \quad z = kx + ly + rt,$$

into Eq.(3) to convert it to the ODE,

$$P(w, w', w'', w''', \cdots) = 0, \tag{4}$$

where P is a polynomial of w and its derivatives, while $' := \frac{d}{dz}$.

Step 2. Suppose that Eq.(4) has the exact solutions as follows:

$$w(z) = \sum_{j=0}^{n} B_j(\exp(-\varphi(z)))^j, \tag{5}$$

where B_j, $(0 \leq j \leq n)$ are constants to be determined latter, such that $B_n \neq 0$ and $\varphi = \varphi(z)$ satisfies the ODE as below:

$$\varphi'(z) = \gamma + \exp(-\varphi(z)) + \mu \exp(\varphi(z)). \tag{6}$$

Eq.(6) has the solutions as follows:
When $\gamma^2 - 4\mu > 0$, $\mu \neq 0$,

$$\varphi(z) = \ln\left(\frac{-\sqrt{(\gamma^2 - 4\mu)}\tanh(\frac{\sqrt{\gamma^2-4\mu}}{2}(z + a)) - \gamma}{2\mu}\right), \tag{7}$$

$$\varphi(z) = \ln\left(\frac{-\sqrt{(\gamma^2 - 4\mu)}\coth(\frac{\sqrt{\gamma^2-4\mu}}{2}(z + a)) - \gamma}{2\mu}\right). \tag{8}$$

When $\gamma^2 - 4\mu < 0$, $\mu \neq 0$,

$$\varphi(z) = \ln\left(\frac{\sqrt{(4\mu - \gamma^2)}\tan(\frac{\sqrt{(4\mu-\gamma^2)}}{2}(z + a)) - \gamma}{2\mu}\right), \tag{9}$$

$$\varphi(z) = \ln\left(\frac{\sqrt{(4\mu - \gamma^2)}\cot(\frac{\sqrt{(4\mu-\gamma^2)}}{2}(z + a)) - \gamma}{2\mu}\right). \tag{10}$$

When $\gamma^2 - 4\mu > 0$, $\gamma \neq 0$, $\mu = 0$,

$$\varphi(z) = -\ln\left(\frac{\gamma}{\exp(\gamma(z+a)) - 1}\right). \tag{11}$$

When $\gamma^2 - 4\mu = 0$, $\gamma \neq 0$, $\mu \neq 0$,

$$\varphi(z) = \ln\left(-\frac{2(\gamma(z+a)+2)}{\gamma^2(z+a)}\right). \tag{12}$$

When $\gamma^2 - 4\mu = 0$, $\gamma = 0$, $\mu = 0$,

$$\varphi(z) = \ln(z+a). \tag{13}$$

Where a is an arbitrary constant and $B_n \neq 0, \gamma, \mu$ are constants in Eq.(7)-Eq.(13). We determine the positive integer n through considering the homogeneous balance between highest order derivatives and nonlinear terms of Eq.(4).

Step 3. Inserting Eq.(5) into Eq.(4) and then considering the function $\exp(-\varphi(z))$ yields a polynomial of $\exp(-\varphi(z))$. Let the coefficients of same power about $\exp(-\varphi(z))$ equal to zero, then we get a set of algebraic equations. We solve the algebraic equations to obtain the values of $B_n \neq 0, \gamma, \mu$ and then we put these values into Eq.(5) along with Eq.(7)-Eq.(13) to finish the determination of the solutions for the given PDE.

3 Exact solutions of the (3+1)-dimensional gBLMP equation

Substitute traveling wave transform

$$u(x, y, s, t) = w(z), \quad z = kx + ly + \delta s + \omega t,$$

into Eq.(1), we get

$$k^3 l w'''' - 6k^2 l w' w'' - 2l\omega w'' + 3l\delta w'' = 0. \tag{14}$$

Integrating it with respect to z and letting the integral constant to be zero, yields

$$k^3 l w''' - 3k^2 l (w')^2 + l(3\delta - 2\omega) w' = 0. \tag{15}$$

Taking the homogeneous balance between w''' and $(w')^2$ in Eq.(15), we have

$$w(z) = B_0 + B_1 \exp(-\varphi(z)), \tag{16}$$

where $B_1 \neq 0$, B_0 are constants.

Substituting $w''', (w')^2, w'$ into Eq.(15) and equating the coefficients of $\exp(-\varphi(z))$ to zero, we get

$$-3\,k^2 l {B_1}^2 - 6\,k^3 l B_1 = 0,$$

$$-6\,{B_1}^2 k^2 l \gamma - 12\,B_1\,k^3 l \gamma = 0,$$

$$-B_1\,k^3l\gamma^3-6\,{B_1}^2k^2l\gamma\,\mu-8\,B_1\,k^3l\gamma\,\mu-3\,B_1\,\delta\,l\gamma+2\,B_1\,l\gamma\,\omega=0,$$
$$-3\,{B_1}^2k^2l\gamma^2-7\,k^3lB_1\,\gamma^2-6\,{B_1}^2k^2l\mu-8\,B_1\,k^3l\mu-3\,lB_1\,\delta+2\,lB_1\,\omega=0.$$
$$-k^3lB_1\,\gamma^2\mu-3\,k^2l{B_1}^2\mu^2-2\,k^3lB_1\,\mu^2-3\,lB_1\,\delta\,\mu+2\,lB_1\,\omega\,\mu=0.$$

Solving the above algebraic equations yields

$$\omega=\frac{1}{2}k^3\gamma^2-2\,k^3\mu+\frac{3}{2}\delta,$$
$$B_1=-2k,$$
$$B_0=\beta, \tag{17}$$

where γ, μ and β are arbitrary constants.

We substitute Eqs.(17) into Eq.(16), then

$$w(z)=\beta-2k\exp(-\varphi(z)). \tag{18}$$

Using Eq.(7) to Eq.(13) into Eq.(18) respectively, we gain exact solutions to the gBLMP equation in the following.

When $\gamma^2-4\mu>0$, $\mu\neq 0$,

$$w_1(z)=\beta+\frac{4k\mu}{\sqrt{(\gamma^2-4\mu)}\tanh(\frac{\sqrt{\gamma^2-4\mu}}{2}(z+a))+\gamma},$$

$$w_2(z)=\beta+\frac{4k\mu}{\sqrt{(\gamma^2-4\mu)}\coth(\frac{\sqrt{\gamma^2-4\mu}}{2}(z+a))+\gamma}.$$

When $\gamma^2-4\mu<0$, $\mu\neq 0$,

$$w_3(z)=\beta-\frac{4k\mu}{\sqrt{(4\mu-\gamma^2)}\tan(\frac{\sqrt{4\mu-\gamma^2}}{2}(z+a))-\gamma},$$

$$w_4(z)=\beta-\frac{4k\mu}{\sqrt{(4\mu-\gamma^2)}\cot(\frac{\sqrt{4\mu-\gamma^2}}{2}(z+a))-\gamma}.$$

When $\gamma^2-4\mu>0$, $\gamma\neq 0$, $\mu=0$,

$$w_5(z)=\beta-\frac{2k\gamma}{\exp(\gamma(z+a))-1}.$$

When $\gamma^2-4\mu=0$, $\gamma\neq 0$, $\mu\neq 0$,

$$w_6(z)=\beta+\frac{k\gamma^2(z+a)}{\gamma(z+a)+2}.$$

When $\gamma^2-4\mu=0$, $\gamma=0$, $\mu=0$,

$$w_7(z)=\beta-\frac{2k}{z+a}.$$

4 Computer simulations

In this section, the computer simulations are given to illustrate our results.

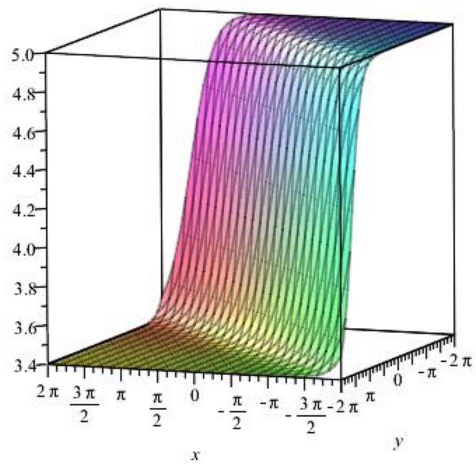

Fig. 1 3D profile of $w_1(z)$ for $\beta = 1$, $k = 1$, $l = 1$, $\delta = 0$, $t = 1$, $\omega = 2$, $a = -1$, $\gamma = 4$, and $\mu = 3$.

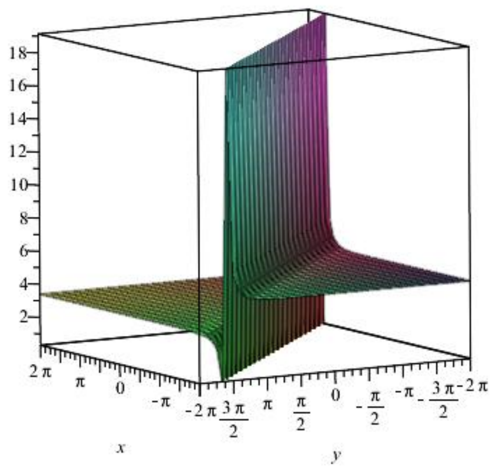

Fig. 2 3D profile of $w_2(z)$ for $\beta = 1$, $k = 1$, $l = 1$, $\delta = 0$, $t = 1$, $\omega = 2$, $a = -1$, $\gamma = 4$, and $\mu = 3$.

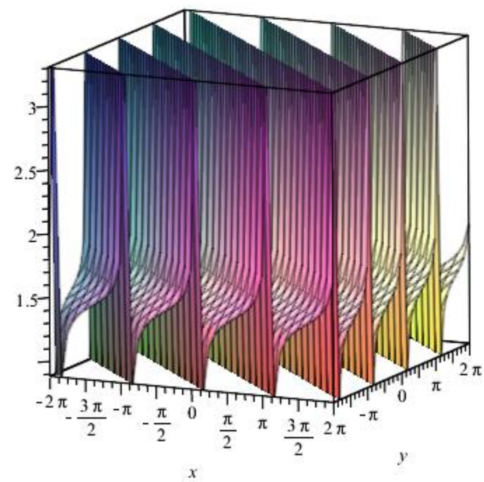

Fig. 3 3D profile of $w_3(z)$ for $\beta = 1$, $k = \frac{1}{10}$, $l = 1$, $\delta = 0$, $t = 1$, $\omega = 2$, $a = -1$, $\gamma = 4$, and $\mu = 5$.

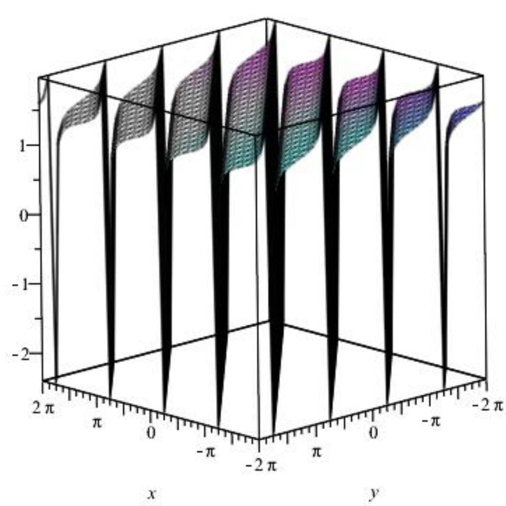

Fig. 4 3D profile of $w_4(z)$ for $\beta = 1$, $k = \frac{1}{10}$, $l = 1$, $\delta = 0$, $t = 1$, $\omega = 2$, $a = -1$, $\gamma = 4$, and $\mu = 5$.

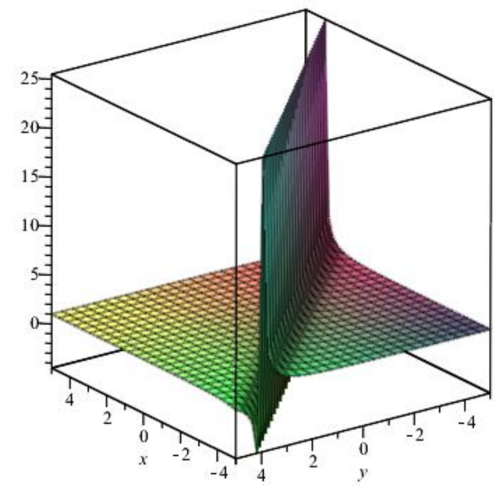

Fig. 5 3D profile of $w_5(z)$ for $\beta = 1$, $k = 1$, $l = 1$, $\delta = 0$, $t = 1$, $\omega = 2$, $a = -1$, and $\gamma = 1$.

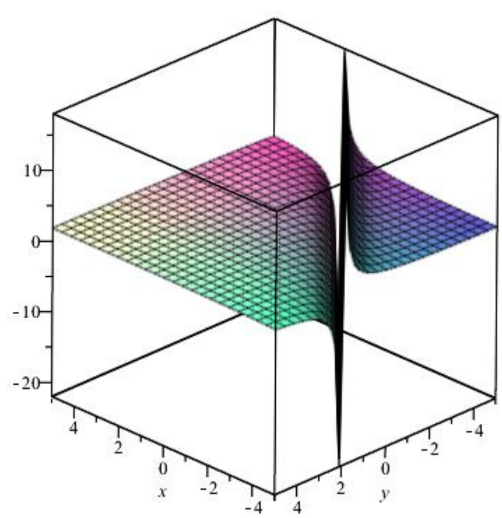

Fig. 6 3D profile of $w_6(z)$ for $\beta = 1$, $k = 1$, $l = 1$, $\delta = 0$, $t = 1$, $\omega = 2$, and $a = -1$.

4 Conclusions

In this paper, using $\exp(-\varphi(z))$-expansion method, we obtain four kinds of exact solutions to the (3+1)-dimensional gBLMP equation including hyperbolic, exponential, trigonometric and rational function solutions. The results show that the applied method is efficient and direct method.

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