n-fold implicative pseudo valuations on hoops

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Abstract

Hoops play an important role in the study of fuzzy logic based on t-norms. In this paper, we introduce some notions of n-fold implicative pseudo valuations on hoops, and also analysis some properties of them. The shrinkage property for n-fold implicative pseudo valuations is provided, and the preimage and image of n-fold implicative pseudo valuation are discussed.

Keywords: Hoop; Pseudo valuation; n-fold implicative pseudo valuation

1. Introduction

A continuous t-norm is a continuous map $*$ from $[0,1]^2$ into $[0,1]$ such that $([0,1],*,1)$ is a commutative totally ordered monoid. Since the natural ordering on $[0,1]$ is a complete lattice ordering, each t-norm induces naturally a residuation, or an implication in more logical. One of the relevant algebraic aspects of a continuous t-norm on $[0,1]$ is the fact that the associated monoid is residuated. Hoops as ordered commutative residuated integral monoids satisfying a further conditions, were introduced by Bosbach [1]. Hoops have long been considered of interest by algebraists, starting from the classical example of the lattice-ordered monoid. Kondo [2] considered that fundamental properties of filters in hoops, and then pointed out that any positive filter of a hoop is implicative and fantastic. To extend the research to filter theory of hoops, [4] introduced the notions of n-fold (positive) implicative filters, [3] gave the notions of some types of filters ((positive) implicative filters, fantastic filters, associative filters) in pseudo hoop-algebras and investigated their properties.

Yang and Xin applied the notion of pseudo-valuations of [5] to EQ-algebras, and studied some characterizations of pseudo pre-valuations on EQ-algebras [6]. They also introduced the notion of pseudo MV-valuations by a function from a BL-algebra to an MV-algebra, which provides a new idea for the study of BL-algebras from MV-algebras [7]. Following the research work of [8], Wang et al. [9] introduced the notion of implicative pseudo valuations on hoops, and showed that a pseudo valuation on regular hoops is implicative if and only if it satisfies $\varphi(x \sqcup x') = 0$.

Considering that the notions of pseudo-valuations [6, 9] and n-fold implicative filters [4], we present the notion of n-fold implicative pseudo valuations on hoops. Some properties of n-fold implicative pseudo valuations are given and the shrinkage property for n-fold implicative pseudo valuations is valid. The preimage and image of n-fold implicative pseudo valuation are also discussed.

2. Preliminaries

By a hoop-algebra or briefly hoop, we shall mean an algebra $(H,\otimes,\rightarrow,1)$ of type $(2,2,0)$ satisfying the following axioms: for any $x,y,z \in H$,

(HP1) $(H,\otimes,1)$ is a commutative monoid;
(HP2) $x \rightarrow x = 1$;
(HP3) $x \otimes (x \rightarrow y) = y \otimes (y \rightarrow x)$;

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(HP4) $x \to (y \to z) = (x \otimes y) \to z$.

On every hoop $(H, \otimes, \to, 1)$, there is a natural order "$\leq$" called the hoop-ordering defined by $x \leq y$ if and only if $x \to y = 1$ for any $x, y \in H$. Under this order, it can be proved that $(H, \leq)$ is a meet semilattice with $x \wedge y = x \otimes (x \to y)$ and 1 as the maximal element. In this work, unless mentioned otherwise, $(H, \otimes, \to, 1)$ will be a hoop, which will often be referred by its support set H.

Proposition 2.1. [10, 11] *Let $(H, \otimes, \to, 1)$ be a hoop. Then the following assertions are valid: for any $x, y, z \in H$,*

(1) $x \otimes y \leq z$ *if and only if* $x \leq y \to z$;
(2) $x \otimes (x \to y) \leq y$, $x \otimes y \leq x \wedge y \leq x \to y$, $x \leq y \to x$;
(3) $x \to y \leq (y \to z) \to (x \to z)$, $y \to x \leq (z \to y) \to (z \to x)$;
(4) $(x \to y) \to (x \to z) \leq x \to (y \to z)$;
(5) $x \to (y \to z) = (x \otimes y) \to z = y \to (x \to z)$;
(6) *if* $x \leq y$, *then* $y \to z \leq x \to z$, $z \to x \leq z \to y$ *and* $x \otimes z \leq y \otimes z$.

Let $(H, \otimes, \to, 1)$ be a hoop and F a nonempty subset of H. F is called a filter if it satisfies: for any $x, y \in H$, (1) $x, y \in F$ implies $x \otimes y \in F$; (2) $x \in F$ and $x \leq y$ imply $y \in F$. It is shown that a nonempty subset F of a hoop H is a filter if and only if for any $x, y \in H$, (1) $1 \in F$; (2) $x \in F$ and $x \to y \in F$ imply $y \in F$. Moreover, a non-empty set F of H is called an implicative filter of H if it satisfies that $x \to (y \to z) \in F$ and $x \to y \in F$ imply $x \to z \in F$, for any $x, y, z \in H$ [12].

We denote $x^n = \underbrace{x \otimes \cdots \otimes x}_{n \text{ times}}$ if $n > 0$ and $x^0 = 1$ for any $x \in H$.

Definition 2.2. [4] *Let F be a subset of H and $n \in N$. F is called a n-fold implicative filter of H if it satisfies:*

(1) $1 \in F$,
(2) $x^n \to (y \to z) \in F$ *and* $x^n \to y \in F$ *imply* $x^n \to z \in F$, *for any* $x, y, z \in H$.

Definition 2.3. [6] *Let $\varphi : H \to R$ be a real-valued function, where R is the set of all real numbers. Then φ is called a pseudo valuation on A with respective a filter if it satisfies the following conditions: for any $x, y \in H$,*

(1) $\varphi(1) = 0$,
(2) $\varphi(y) \leq \varphi(x) + \varphi(x \to y)$.

A pseudo valuation φ is called a valuation if $\varphi(x) = 0$ implies $x = 1$.

Proposition 2.4. [6] *Let φ be a pseudo valuation on H. Then the following inequalities are valid: for any $x, y, z \in H$,*

(1) $x \leq y$ *implies* $\varphi(y) \leq \varphi(x)$,
(2) $0 \leq \varphi(x)$,
(3) $\varphi(x \to z) \leq \varphi(x \to y) + \varphi(y \to z)$,
(4) $\varphi(x \to (y \to z)) \leq \varphi((x \to y) \to z)$.

Definition 2.5. [9] *A real-valued function φ on H is called an implicative pseudo valuation if it satisfies:*

(1) $\varphi(1) = 0$,
(2) $\varphi(x \to z) \leq \varphi(x \to (y \to z)) + \varphi(x \to y)$, *for any* $x, y \in H$.

Proposition 2.6. [9] *Every implicative pseudo valuation on H is a pseudo valuation on H.*

Definition 2.7. *Let H_1, H_2 be Hoops. A function $f : H_1 \to H_2$ is called a hoop-homomorphism if*

(1) $f(1) = 1$,
(2) $f(a \otimes b) = f(a) \otimes f(b)$,
(3) $f(a \to b) = f(a) \to f(b)$,

for any $a, b \in H_1$.

3. *n*-fold implicative pseudo valuations

In the section, the notion of pseudo valuations on hoop-algebras is given, and some characterizations of pseudo valuations are shown.

Definition 3.1. *Let φ be a real-valued function on H and $n \in N$. Then φ is called a n-fold implicative pseudo valuation on H if it satisfies:*

(1) $\varphi(1) = 0$,
(2) $\varphi(x^n \to z) \leq \varphi(x^n \to (y \to z)) + \varphi(x^n \to y)$, *for any* $x, y \in H$.

Remark 3.2. (1) *Notice that* 1*-fold implicative pseudo valuation on a hoop is an implicative pseudo valuation.*
(2) *The notion of n-fold implicative pseudo valuations on a hoop generalizes the notion of implicative pseudo valuations.*
(3) *Every n-fold implicative pseudo valuations on a hoop is a pseudo valuation.*

The following example shows that n-fold implicative pseudo valuations are exist.

Example 3.3. *Let $H = \{0, a, b, c, 1\}$ be a set with the Hasse diagram and Cayley tables as follows.*

⊗	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	a	b
1	0	a	b	1

→	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

Then $(H, \otimes, \to, 1)$ is a hoop. Define a real-valued function $\varphi : H \to R$ by $\varphi(0) = 5$, $\varphi(a) = 2$, $\varphi(b) = 3$ and $\varphi(1) = 0$. Then φ is a n-fold implicative pseudo valuation on H, while it is not a 2-fold implicative pseudo valuation, since $\varphi(b^2 \to 0) = 3 \nleq \varphi(b^2 \to (b^2 \to 0)) + \varphi(b^2 \to b^2) = 0$.

Proposition 3.4. *Let φ be a real-valued function of H. If φ is a n-fold implicative pseudo valuation on H, then the set $H_\varphi := \{x \in H | \varphi(x) = 0\}$ is a n-fold implicative filter of H.*

Proof. Obviously, $1 \in H_\varphi$. For any $x^n \to (y \to z) \in H_\varphi$ and $x^n \to y \in H_\varphi$, then we have $\varphi(x^n \to (y \to z)) = 0$ and $\varphi(x^n \to y) = 0$. Notice that φ is a n-fold implicative pseudo valuation, we obtain that $\varphi(x^n \to z) \leq \varphi(x^n \to (y \to z)) + \varphi(x^n \to y) = 0$, and $\varphi(x^n \to z) \geq 0$. Hence $\varphi(x^n \to z) = 0$, it follows that $x^n \to z \in H_\varphi$, and so H_φ is a n-fold implicative filter of H.

Theorem 3.5. *Let φ be a pseudo valuation on H. Then the following conditions are equivalent: for any $x, y, z \in H$,*

(1) *φ is a n-fold implicative pseudo valuation on H,*
(2) $\varphi(x^n \to y) \leq \varphi(x^{n+1} \to y)$,
(3) $\varphi(x^n \to x^{2n}) = 0$,
(4) $\varphi((x^n \to y) \to (x^n \to z)) \leq \varphi(x^n \to (y \to z))$.

Proof. (1) ⇒ (2) For any $x, y \in H$, we get that $x^n \to x = 1$, and $\varphi(x^n \to y) \leq \varphi(x^n \to (x \to y)) + \varphi(x^n \to x) = \varphi(x^{n+1} \to y) + \varphi(1) = \varphi(x^{n+1} \to y)$, hence (2) holds.

(2) ⇒ (3) The proof is by induction on n. Suppose that (2) holds.

Firstly, for $n = 1$, $\varphi(x \to x^2) \leq \varphi(x^{1+1} \to x^2) = 0$, we have $\varphi(x \to x^2) = 0$.

Secondly, for $n = 2$, then $\varphi(x^3 \to x^4) = \varphi(x^2 \to (x \to x^4)) \leq \varphi(x^3 \to (x^x \to x^4)) = \varphi(1) = 0$, hence $\varphi(x^3 \to x^4) = 0$. From $\varphi(x^2 \to x^4) \leq \varphi(x^3 \to x^4) = 0$, we get $\varphi(x^2 \to x^4) = 0$.

Finally, for $n > 2$, from $x^{n+1} \to (x^{n-1} \to x^{2n}) = 1$, we obtain that $\varphi(x^n \to (x^{n-1} \to x^{2n})) \leq \varphi(x^{n+1} \to (x^{n-1} \to x^{2n})) = 0$, and so $\varphi(x^n \to (x^{n-1} \to x^{2n})) = 0$, that is $\varphi(x^{n-1} \to (x^n \to x^{2n})) = 0$. By using the hypothesis n times, then we get $\varphi(x^{n-n} \to (x^n \to x^{2n})) = 0$, and so $\varphi(x^n \to x^{2n}) = 0$.

(3) ⇒ (4) For any $x, y, z \in H$, we have $x^n \to (y \to z) \leq x^n \to ((x^n \to y) \to (x^n \to z)) = x^n \to (x^n \to ((x^n \to y) \to z)) = x^{2n} \to ((x^n \to y) \to z) \leq (x^n \to x^{2n}) \to (x^n \to ((x^n \to y) \to z)) = (x^n \to x^{2n}) \to$

$(x^n \to ((x^n \to y) \to z))$. Since φ is a pseudo valuation on H and $\varphi(x^n \to x^{2n}) = 0$, it follows that $\varphi((x^n \to y) \to (x^n \to z)) \leq \varphi(x^n \to x^{2n}) + \varphi((x^n \to x^{2n}) \to (x^n \to ((x^n \to y) \to z))) = \varphi((x^n \to x^{2n}) \to (x^n \to ((x^n \to y) \to z))) \leq \varphi(x^n \to (y \to z))$, which means that $\varphi((x^n \to y) \to (x^n \to z)) \leq \varphi(x^n \to (y \to z))$.

(4) $\Rightarrow$ (1) Since φ is a pseudo valuation on H, we get that $\varphi(x^n \to z) \leq \varphi(x^n \to y) + \varphi((x^n \to y) \to (x^n \to z)) \leq \varphi(x^n \to y) + \varphi(x^n \to (y \to z))$, therefore φ is a n-fold implicative pseudo valuation on H.

Proposition 3.6. *If φ is a n-fold implicative pseudo valuation on H, then $\varphi(x^n \to y) = \varphi(x^{n+1} \to y)$ for any $x, y \in H$.*

Proof. According to Theorem 3.5, we get $\varphi(x^n \to y) \leq \varphi(x^{n+1} \to y)$. As for the reverse inequality, from $x^n \to y \leq x^{n+1} \to y$, we have $\varphi(x^{n+1} \to y) \leq \varphi(x^n \to y)$ by Proposition 2.4. Thus $\varphi(x^n \to y) = \varphi(x^{n+1} \to y)$.

From Theorem 3.5, if φ is a n-fold implicative pseudo valuation on H, then $\varphi((x^n \to y) \to (x^n \to z)) \leq \varphi(x^n \to (y \to z))$. And notice that $(x^n \to y) \to (x^n \to z) \leq x^n \to (y \to z)$ by Proposition 2.1 (6), we have $\varphi(x^n \to (y \to z)) \leq \varphi((x^n \to y) \to (x^n \to z))$, so we get the following result.

Proposition 3.7. *If φ is a n-fold implicative pseudo valuation on H, then $\varphi((x^n \to y) \to (x^n \to z)) = \varphi(x^n \to (y \to z))$ for any $x, y, z \in H$.*

Lemma 3.8. *Every n-fold implicative pseudo valuation φ on H is a $(n + 1)$-fold implicative pseudo valuation.*

Proof. If φ is a n-fold implicative pseudo valuation on H, then $\varphi(x^{n+1} \to y) = \varphi(x^n \to (x \Rightarrow y)) \leq \varphi(x^{n+1} \to (x \to y)) = \varphi(x^{n+2} \to y$, that is, $\varphi(x^{n+1} \to y) \leq \varphi(x^{n+2} \to y$, hence φ is a $(n + 1)$-fold implicative pseudo valuation by Theorem 3.5.

Using Lemma 3.8 and a simple induction argument, we obtain the following proposition.

Proposition 3.9. *Let φ be a real-valued function of H and $k \in N - \{0\}$. If φ is a n-fold implicative pseudo valuation on H, then φ is $(n + k)$-fold implicative pseudo valuation.*

In the follows, we will show that the shrinkage property for n-fold implicative pseudo valuations on a hoop is valid.

Proposition 3.10. *Let φ be a real-valued function on H and ψ be a pseudo valuation on H with $\psi \leq \varphi$, that is, $\psi(x) \leq \varphi(x)$ for any $x \in H$. If φ is a n-fold implicative pseudo valuation on H, then ψ is also a n-fold implicative pseudo valuation on H.*

Proof. Since φ is a n-fold implicative pseudo valuation on H, then $\psi(x^n \to x^{2n}) \leq \varphi(x^n \to x^{2n}) = 0$ for any $x \in H$. Consider that ψ is a pseudo valuation on H, we get that $\psi(x^n \to x^{2n}) \geq 0$ by Proposition 2.4, and therefore $\psi(x^n \to x^{2n}) = 0$, hence ψ is a n-fold implicative pseudo valuation on H.

Definition 3.11. *Let f be a mapping from an hoop H_1 into a hoop H_2, and φ, ψ be real-valued function on H_1 and H_2, respectively. Then*

(1) *the preimage $f^{-1}(\psi)$ of H_2 under f is defined as $f^{-1}(\psi)(x) = \psi(f(x))$, for any $x \in H_1$;*
(2) *the image $f(\varphi)$ of φ under f is defined as*

$$f(\varphi)(y) = \begin{cases} \inf\{\varphi(x) | f(x) = y\}, & f^{-1}(y) \neq \emptyset, \\ 0, & \textit{otherwise}. \end{cases}$$

Theorem 3.12. *Let φ, ψ be n-fold implicative pseudo valuations on H_1 and H_2, respectively.*

(1) *If $f : H_1 \to H_2$ be a hoop-homomorphism, then the preimage $f^{-1}(\psi)$ is a n-fold implicative pseudo valuation on H_1.*
(2) *If f is a hoop-epimorphism, then the image $f(\varphi)$ is a n-fold implicative pseudo valuation on H_2.*

Proof. It is easy to prove that $f^{-1}(\psi)$ and $f(\varphi)$ are pseudo valuations on H_1 and H_2, respectively.

(1) $f^{-1}(\psi)(x^n \to x^{2n}) = \psi(f(x^n \to x^{2n})) = \psi(f(x)^n \to f(x)^{2n}) = 0$, hence $f^{-1}(\psi)$ is a n-fold implicative pseudo valuation on H_1.

(2) Since φ is a n-fold implicative pseudo valuations on H_1 and f is a hoop-epimorphism, For any $y \in H_2$, then there exists $x \in H_1$ such that $f(x) = y$. It follows that $f(\varphi)(y^n \to y^{2n}) = \inf\{\varphi(z) | f(z) = y^n \to y^{2n}, z \in H_1\} = \inf\{\varphi(z) | f(z) = f(x)^n \to f(x)^{2n}, z \in H_1\} = \inf\{\varphi(z) | f(z) = f(x^n \to x^{2n}), z \in H_1\} = 0$, and hence $f(\varphi)$ is a n-fold implicative pseudo valuation on H_2.

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