A Note on The Mathematical Beauty of The Gamma Function

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Abstract: *In this research paper, we will investigate some applications and properties of the Gamma Function. We will discuss and illustrate some of the beautiful applications of the Gamma Function in mathematics and statistics through different examples.*

Introduction: In 1729, Leonhard Euler attempted to expand the domain of the factorial to the real numbers. He derived the integral notation for $n!$, which lead to the beginning of the Gamma function. In today's world, the Gamma function is encountered often in mathematics and natural sciences and is applicable in discrete mathematics, applied sciences, statistics, and engineering. In order to understand the working of the Gamma function we must first understand what exactly the Gamma function is. The Gamma function, $\Gamma(n)$, is defined to be an extension of the factorial to complex and real number arguments. There are two forms of the generalized factorial, both of which were derived by Euler. The forms are as follows:

$$n! = \prod_{k=1}^{\infty} \frac{\left(1+\frac{1}{k}\right)^{n}}{1+\frac{n}{k}} \quad and \quad n! = \int_{0}^{\infty} (-\log s)^{n}\, ds \quad for \;\; n > 0$$

For now, we will refer to these as the product and integral definitions. In the 19th century, Carl Friedrich Gauss rewrote Euler's product definition to extend the domain to the complex plane, rather than simply the real numbers. This expression uses a limiting process on a series of intermediate functions to represent the factorial.

$$\Gamma_r(x) = \frac{r!\, r^{x}}{x(1+x)(2+x)\ldots(r+x)}$$

$$\Gamma(x) = \lim \Gamma_r(x)$$

This was during the same time period that Adrien-Marie Legendre suggested the notation Γ and also transformed the integral definition in a way to extend its domain:

$$n! = \int_0^1 (-\log s)^n \, ds$$

$$let\ t = -\log s$$

$$s = -e^t$$

$$dt = -\frac{1}{s} ds = e^t ds$$

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-ct} dt,$$

which is the more common integral definition that is seen and used today. In 1922, Bohr and Mollerup proved that the Gamma function is the only function that satisfies the recurrence relationship and is logarithmically convex. Their results are summarized in the following theorem: We sate the theorem with proof and apply it.

Theorem 1 : (Known as the Bohr Moller up Theorem).

If, $\Gamma_1 = \Gamma_2$ on the integers and $\Gamma_1(\boldsymbol{x}) = \Gamma_2(\boldsymbol{x})$ is logarithmically convex for $\boldsymbol{Re}(\boldsymbol{x}) > \mathbf{0}$, then $\Gamma_1 = \Gamma_2$ $\forall \boldsymbol{x} \in \mathfrak{R}$. Given this theorem, it is important to be sure that the Gamma function satisfies the criteria for log convexity. This can be done using the proposition that $\Gamma(\boldsymbol{x})$ is logarithmically convex for all $\boldsymbol{x} > \mathbf{0}$.

Main Results

The main results of this paper will not only inform the reader about what the Gamma function is, but also show and explain the properties and the applications of the Gamma function..

Theorem 2: The Fundamental Identity.

$$\Gamma(\mathbf{k}+\mathbf{1}) = \mathbf{k}\Gamma(\mathbf{k}) \quad \mathbf{for} \quad \mathbf{k} > \mathbf{0}$$

Proof: This follows from integrating by parts, with $\mathbf{u} = \mathbf{x^k}$ and $\mathbf{dv} = \mathbf{e^{-x}dx}$:

$$\Gamma(\mathbf{k}+\mathbf{1}) = \int_0^\infty \mathbf{x^k e^{-x} dx} = (-\mathbf{x^k e^{-x}})_0^\infty + \int_0^\infty k x^{k-1} e^{-x} dx = \mathbf{0} + k\Gamma(k)$$

Applying this result repeatedly gives

$$\Gamma(\mathbf{k}+\mathbf{n}) = (\mathbf{k}(\mathbf{k}+\mathbf{1})\cdots(\mathbf{k}+\mathbf{n}-\mathbf{1})\Gamma(\mathbf{k}),$$

It's clear that the Gamma function is a continuous extension of the factorial function. Another property of the Gamma function is the property of recurrence. The Gamma function satisfies the recursive property

$$\Gamma(\boldsymbol{\alpha}) = (\boldsymbol{\alpha} = \mathbf{1})\Gamma(\boldsymbol{\alpha} - \mathbf{1})$$

which can be proved using integration by parts and L'Hopital's Rule. Note that the recursive relationship in the Gamma function can be used to extend the definition of the function to all positive real numbers. When $a=n$ and n is a positive integer, then the Gamma function is related to the factorial function:

$$\Gamma(\boldsymbol{n}) = (\boldsymbol{n} - \mathbf{1})!$$

Remark 1. For specific values of a, exact values of $\Gamma(\mathbf{a})$ exist. For the Gamma function evaluated at $\boldsymbol{\alpha} = \frac{\mathbf{1}}{\mathbf{2}}$ it follows that:

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

This recursive relationship can be used to calculate the value of the Gamma function for all positive real numbers by knowing the value of the Gamma function between 1 and 2. The following table contains the Gamma function for arguments between 1 and 1.99.

***<u>Table 1</u>*.**

$$\Gamma(\mathbf{x}), \qquad \mathbf{1} \leqq \mathbf{x} \leqq \mathbf{1.99}$$

x	0	1	2	3	4	5	6	7	8	9
1.0	1.0000	.9943	.9888	.9835	.9784	.9735	.9687	.9642	.9597	.9555
.1	.9514	.9474	.9436	.9399	.9364	.9330	.9298	.9267	.9237	.9208
.2	.9182	.9156	.9131	.9108	.9085	.9064	.9044	.9025	.9007	.8990
.3	.8975	.8960	.8946	.8934	.8922	.8912	.8902	.8893	.8885	.8879
.4	-8873	.8868	.8864	.8860	.8858	.8857	.8856	.8856	.8857	.8857
.5	.8862	.8866	.8870	.8876	.8882	.8889	.8896	.8905	.8914	.8924
.6	.8935	.8947	.8959	.8972	.8986	.9001	.9017	.9033	.9050	.9068
.7	.9086	.9106	.9126	.9147	.9168	.9191	.9214	.9238	.9262	.9288
.8	.9314	.9341	.9368	.9397	.9426	.9456	.9487	.9518	.9551	.9584
.9	.9618	.9652	.9688	.9724	.9761	.9799	.9837	.9877	.9917	.9958

Remark 2. *Let* α=*x*. To explain, the following examples will show how to evaluate the Gamma function for positive integers and fractions and negative non-integer values. First, suppose that $\boldsymbol{\alpha}$ equals that positive integer $\boldsymbol{5}$. Then it follows that:

$$\Gamma(\mathbf{5}) = \mathbf{4}\Gamma(\mathbf{4}) = \mathbf{4}\cdot\mathbf{3}\Gamma(\mathbf{3}) = \mathbf{4}\cdot\mathbf{3}\cdot\mathbf{2}\cdot\mathbf{1}\cdot\Gamma(\mathbf{1}) = \mathbf{4}! = \mathbf{24}$$

Next, suppose that $\boldsymbol{\alpha} = \frac{\mathbf{23}}{\mathbf{3}}$, a positive fraction. Then it follows that:

$$\Gamma\left(\frac{23}{3}\right) = \left(\frac{20}{3}\right)\Gamma\left(\frac{20}{3}\right) = \left(\frac{20}{3}\right)\left(\frac{17}{3}\right)\Gamma\left(\frac{17}{3}\right)$$

$$= \left(\frac{20}{3}\right)\cdot\left(\frac{17}{3}\right)\cdot\left(\frac{14}{3}\right)\cdot\left(\frac{11}{3}\right)\cdot\left(\frac{8}{3}\right)\cdot\left(\frac{5}{3}\right)\Gamma\left(\frac{5}{3}\right)$$

$$= \frac{2094400}{729}\Gamma\left(\frac{5}{3}\right)$$

$$= \frac{2094400}{729}\Gamma(1.67)$$

$$= \frac{2094400}{729}\Gamma(.90330)$$

=2595.158

Note: Use table one, above, to find the value for Γ(1.67)

Lastly, for negative non-integer values, suppose that $\alpha = -\frac{5}{6}$. Substituting $\alpha + 1$ for α in the function for the recursive property, yields

$$\boldsymbol{\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)} \ .$$

Solving now for $\Gamma(\alpha)$ yields

$$\Gamma(\boldsymbol{\alpha}) = \frac{\Gamma(\boldsymbol{\alpha}+\mathbf{1})}{\boldsymbol{\alpha}}.$$

$\boldsymbol{\alpha}$ Therefore, $\Gamma\left(-\frac{\mathbf{5}}{\mathbf{6}}\right)$ is,

Using linear interpolation between **1.16** and **1.17** yields $\Gamma\left(\frac{7}{6}\right) = \mathbf{.927733333333}$.

Hence,

$\Gamma\left(-\frac{\mathbf{5}}{\mathbf{6}}\right) = \mathbf{-6.67968}$, which is close to the more exact answer of **-6.679579**.

Continuing with its properties, the Gamma function satisfies the following functional equations:

- $\Gamma(\boldsymbol{k}+\mathbf{1}) = \boldsymbol{k}\Gamma(\boldsymbol{k})$ *(mentioned above)*
- $\Gamma(\boldsymbol{n}) = (\boldsymbol{n}-\mathbf{1})!$ *(mentioned above)*

- $\Gamma(1) = 1$
- $\Gamma(1-k)\Gamma(k) = \frac{\pi}{\sin \pi k}$

Remark 3. We express the derivatives of the Gamma function in terms of another special function, the Polygamma function $\psi^{(k)}$.

$$\psi^{(m)}(z) = (-1)^{(m+1)} \int_0^\infty \frac{t^m e^{-zt}}{1-e^{-1}} dt$$

More important in our context is the following form:

$$\psi^{(m)}(z) = \left(\frac{d}{dz}\right)^{m+1} (\log \Gamma(z))$$

The digamma is the first polygamma function, in this case $m = 0$.

$$\psi^{(0)}(z) = \frac{\Gamma'(z)}{\Gamma(z)}.$$

$$\Gamma'(z) = \psi^{(0)}(z)\Gamma(z)$$

This leads us to the following expression for the derivatives of gamma:

$$\Gamma^{(k)}(z) = \int_0^\infty t^{z-1} e^{-t} (\log t)^n dt$$

Since some properties of the Gamma function have now been covered and illustrated, it is time to move on to the applications of the function. As stated in the introduction, the Gamma function has applications in mathematics, applied sciences, and statistics. Beginning with mathematical applications, factorials are used in the study of counting and probability. For example, permutations and combinations both require the use of factorials when the number of objects is large. Since the Gamma function is an extension of the factorial, we can input almost any real or complex number into the function and find its value. In the world of Probability and Statistics, the Gamma function is referred to as the Gamma distribution.

Remark 4. The integral definition of the Gamma distribution-in Statistics-is very similar to that of the Gamma function with only a few exceptions. For example, recall the integral definition for the Gamma function:

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-ct} dt$$

Now, consider the integral

$$\int_{0}^{\infty} \boldsymbol{y^{\alpha-1}\, e^{-y/\beta}\, dt}$$

These are the same integral if $\boldsymbol{y = t}$, $\boldsymbol{\alpha = n}$, and $\boldsymbol{\beta = c^{-1}}$ So, the value of the integral for the Gamma distribution is $\boldsymbol{\beta^{\alpha}}\Gamma(\boldsymbol{\alpha})$. This implies that

$$\frac{\mathbf{1}}{\boldsymbol{\beta^{\alpha}}\Gamma(\boldsymbol{\alpha})}\boldsymbol{y^{\alpha-1}e^{-y/\beta}}$$

is a probability distribution function, or PDF, with support $(0, \infty)$. One of the main applications of the Gamma distribution is based on the interval, which occurs between events when derived from it, becomes the sum of one or more than one exponentially distributed variables. As it is moderately skewed, it can be very well used in different areas. For example, it can be used as a workable model for climatic conditions, like raining, and also in financial services in order to model the different patterns of insurance claims and the different size of loan defaults. Therefore, it has been used in different levels of Probability-Of-Ruin and Value-At-Risk calculations.

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