A note On Fascinating Mathematical Applications Arithmetic & Geometric Sequences

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ABSTRACT: A sequence is a set of numbers in a specific order. The two simplest sequences which are interesting to work with are the classical arithmetic and geometric sequences. Since arithmetic and geometric sequences are so nice and regular, they have simple and friendly formulas. They have many interesting mathematical properties which are enjoyable and have exciting mathematical patterns.

In mathematics, an arithmetic sequence is a sequence of numbers such that the difference of any two successive members of the sequence is a constant called common difference. For instance, the sequence 3, 5, 7, 9, 11, 13... is an arithmetic sequence with common difference 2.

If the initial term of an arithmetic sequence is A_1 and the common difference of successive members is d, then the nth term $\mathbf{A}_n$ of the sequence is given by:

$$\mathbf{A}_n = \mathbf{A_1 + (n - 1)d}$$

Likewise, a geometric sequence is a sequence of numbers where each term after the first is found by multiplying the previous one by a fixed non-zero number called the common ratio. For example, the sequence 2, 6, 18, 54, ... is a geometric sequence with common ratio 3. Similarly 10, 5, 2.5, 1.25, ... is a geometric sequence with common ratio 1/2

If the initial term of a geometric sequence is G_1 and the common ratio r, then the nth term G_n of the sequence is given by:

$$\mathbf{G_n = G_1\, r^{(n - 1)}}$$

These two sequences have many mathematical properties and patterns that are worthy of exploration in today's mathematics world.

In this paper we will make our journey with the fascinating mathematical beauty of these two celebrity sequences. The purpose of the study is to dig out some important results and practical applications concerning the arithmetic and geometric sequences. We deeply examine some of the interesting properties and patterns of these two shining stars of the classical number sequences. Some important theorems dealing with the mathematical concepts of the two sequences will be proved. We also investigate beautiful connections that exist between these sequences and the seemingly unrelated mathematical territories of perfect and triangular numbers. Different impressive problem solving techniques will be shown. Real life application of arithmetic and geometric sequences will be discussed.

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1. Introduction: The arithmetic and geometric sequences are exciting classical sequence with rich applications. They are the easiest examples of simple number sequences. They are the most useful sequences with fascinating properties and patterns. Both the arithmetic and geometric are recursive sequences.

The recursive formula for an arithmetic sequence (A.S.) is written in the form

$$\mathbf{A_n = A_{n-1} + d.}$$

The recursive formula for a geometric sequence (G.S.) is written in the form

$$\mathbf{G_n = rG_{n-1}}$$

Geometric sequences (with common ratio not equal to -1,1 or 0) show exponential growth or exponential decay, as opposed to the linear growth (or decline) of an arithmetic sequence.

The arithmetic and geometric sequences are related; exponentiating each term of an arithmetic sequence yields a geometric sequence , while taking the logarithm of each term in a geometric sequence with a positive common ratio yields an arithmetic sequence. These sequences have very interesting properties and keep popping up in many places as illustrated below.

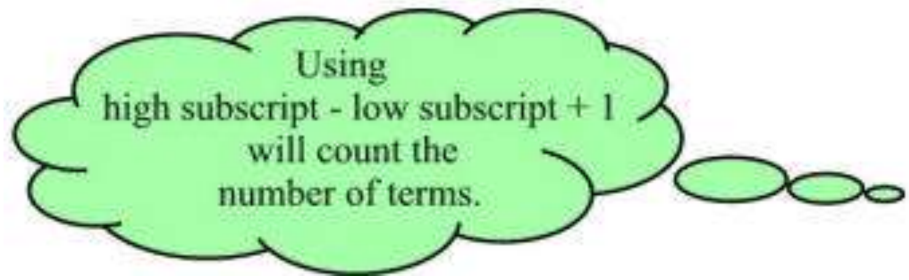
2. Background Materials: The Following facts and main results will be listed here and used as quick references

(a). To find any term of an arithmetic sequence we use**:**

$$\mathbf{A_n = A_1 + (n - 1)d}$$

(b) Generally, to check whether a given sequence is arithmetic one simply checks whether successive entries in the sequence all have the same difference.

(c) Number of terms is: $(A_n - A_1) + 1$ or



(d) An **arithmetic mean** is the term between any two terms of an arithmetic sequence. It is simply the average (mean) of the given terms.

(e) To find any term of geometric sequence we use**:**

$G_n = \mathbf{G_1\ r^{(n-1)}}$.

(f) Generally, to check whether a given sequence is geometric, one simply checks whether successive entries in the sequence all have the same ratio.

(g) The common ratio of a geometric series may be negative, resulting in an alternating sequence, with numbers switching from positive to negative and back.

(h). The behaviors of a geometric and arithmetic sequence depend on the value of the common ratio and common difference.

(i) **Dirichlet's Theorem on Primes in Arithmetic Sequence** (1837)
If A_1 and d are relatively prime positive integers, then the arithmetic sequence A_1, A_1+d, A_1+2d, A_1+3d, ... contains infinitely many primes.

(J) A geometric (arithmetic) series is the *sum* of the numbers in a geometric(arithmetic) sequence.

(k) A positive integer P is called a perfect number if it is equal to the sum of its proper positive divisors [1].

(m) The triangular numbers are numbers formed by partial sum of the arithmetic sequence 1,2,3,4, 5, ….n . In other words, triangular numbers are those counting numbers that can be written as T_n = 1+2+3+…+ n = $\frac{n(n+1)}{2}$ [2]

3. <u>SOME PROBLEM SOLVING TECHNIQUES USING ARITHEMATIC AND GEOMETRIC SEQUENCES</u>

Next we will illustrate some fascinating problem solving techniques applying the properties and patterns of arithmetic and geometric sequences.

Problem 1. Give **a**n example of two arithmetic sequences that contain infinitely many primes.

Solution: This is possible by Dirichlet's Theorem**.** The following are the easy ones to consider.

(a) 1, 7, 13, 19, 25, 31, 37, ...
(b) 5, 11, 17, 23, 29, 35, 41, ...

Note tha**t** together these two sequences seem to contain all of the primes except 2 and 3

Problem 2. Find A_{15} for an arithmetic sequence where

$A_3 = -7 + 3k$ and $A_6 = -15 + 13k$.

Solution: Note that:

$$A_6 = -7+3k+3d$$

$$\Rightarrow -15 + 13k = -7+3k+3d$$

$$\Rightarrow 3d = -8+10k$$

Now , we have $A_{15} = -7+3k+3d$

$$= -7+3k-8+10k$$

$$= -15+13k.$$

Problem 3. Savannah State Herty Hall Conference Room has square tables which seat four people. When two tables are placed together, six people can be seated. If 5000 square tables are placed together to form one very long table, how many people can be seated?

Solution: The pattern that is emerging is clearly an arithmetic **s**equence. The numbers in the sequence begin **4, 6, 8, 10, ...** .

Using $\mathbf{A}_n = A_1 + (n - 1)d$ with $n = 5000$, $d = 2$ and $A_1 = 4$,

we get the 5000th term = 4 + [(5000 - 1) X 2]

= 4 + [4999 X 2]

=4 + 9998

$= 10002$

Therefore, 10002 people could sit at 5000 tables.

<u>Problem 4</u>. A company began doing business four years ago. Its profits for the last 4 years have been \$11 million, \$15 million, \$19, million and \$23 million. If the pattern continues, what is the expected profit in 30 years

Solution: The pattern that is emerging is clearly an arithmetic **s**equence. The numbers in the sequence begin 11**, 15, 19, 23, ..** . Using $\mathbf{A}_n = A_1 + (n - 1)d$ with $n = 30$, $d = 4$ and $A_1 = 11$,

we get the 30th term $= 11 + [(30 - 1) \text{ X } 4]$

$= 11+116$

$= 127$

Therefore, 127 million will be the profit in 30 years.

<u>Problem 5.</u> The third term of a geometric sequence is 3 and the sixth term is 1/9. Find the first term

<u>Solution:</u> **Note we have** $G_6 = 3\ r^3$

$$\Rightarrow \frac{1}{9} = 3\ r^3$$

$$\Rightarrow \frac{1}{3} = r$$

Now using $\mathbf{3 = G_1} \left(\frac{1}{3}\right)^2$**,** we get

$G_1 = 27$

Therefore, the first term is 27.

<u>Problem 6.</u> Find the number of multiples of 9 between 17 and 901.

<u>Solution</u>: Note that using arithmetic sequence formula we have

$900 = 18 + 9(n - 1)$ and solve for n.

$864 = 9n - 9$

$873 = 9n$

$99 = n$

There are 99 multiples in the given range.

Problem 7 . Find the 9th term of the sequence

$$1, \sqrt{2}, 2, \ldots$$

Solution: This is geometric sequence with $r = \sqrt{2}$. We have

$G_9 = 1\,(\sqrt{2})^8.$

$= 2^4$

$=16$

Therefore the 9th term is 16.

Problem 8 Find G_5 for a geometric sequence
where $A_1 = 2 + 2k$ and $r = 3$

Solution $G_5 = (2+2k)\ 3^4.$

$= 81(2+2k)$

Problem 9. A ball is dropped from a height of 100 feet. The ball bounces to 80% of its previous height with each bounce. How high (*to the nearest tenth of a foot*) does the ball bounce on the fifth bounce?

Solution: Setting up a model drawing for each "bounce", we get the geometrics sequence, 80, 64, 51.2, … and r =0.8.

Now using $G_n = G_1\ r^{(n-1)}$, we get

$$G_5 = 80\left(\frac{4}{5}\right)^4$$

$$= 80\left(\frac{256}{625}\right)$$

Problem 10. Three numbers form an arithmetic sequence, the common difference is 11. If the first number is decreased by 6, the second is decreased by 1, and the third number is doubled, the resulting numbers form a geometric sequence. Determine the three numbers that form the geometric sequence.

Solution: Three numbers in A.S. can be written:

A, A + d, A + 2d.

It the d is 11, write;

A, A + 11, A + 22.

If the first number is decreased by 6,

write A - 6

the second is decreased by 1,

write A + 10

and the third number is doubled

write 2A + 44

If three numbers form a geometric sequences. then there is a common ratio:

$$\frac{\text{Second}}{\text{First}} = \frac{\text{Third}}{\text{Second}}$$

$$\frac{A + 10}{A - 6} = \frac{2A + 44}{A + 10}$$

$A^2 + 20A + 100 = 2A^2 + 32A - 264$

$A^2 + 12A - 364 = 0$

$(A + 26)(A - 14) = 0$

$A = -26, A = 14$

If A = - 26, the A.s. is -26,-15,-4 and the G.S. would be:

-32, -16, -8, so your common ratio is 2.

If A = 14, the A.S. is 14, 25, 36 and the G.S. would be:

8, 24, 72, so your common ratio is 3.

4. Main Results

Next we look at some of the important theorems in the mathematics of the arithmetic and geometric sequences

Theorem 1. The sum s_n of the first n terms of an arithmetic sequence is given by

$$S_n = \sum_{k=1}^{n}\left(A_1 + (k-1)d\right) = n/2(A_1 + A_n)$$

Proof: Note that S_n can be expressed in two ways as follows:

(1) $S_n = A_1 + (A_1+d) + (A_1+2d) + (A_1+3d) + \ldots + (A_1+(n-3)d)$

$+ (A_1+(n-2)d) + (A_1+(n-1)d)$

(2) $S_n = (A_n-(n-1)d) + ((A_n-(n-2)d)) + (A_n-(n-3)d) + \ldots + (A_n-2d) + (A_n-d) + A_n$

Now Adding (1) and (2), we get

$$2S_n = n\left(A_1 + A_n\right)$$

$$\Rightarrow S_n = \frac{n\left(A_1 + A_n\right)}{2}$$

Hence, the theorem is proved.

Corollary 1. $S_n = \dfrac{n\left(2A_1 + (n-1)d\right)}{2}$

Proof: The corollary follows from Theorem 1 by using

$\mathbf{A_n = A_1 + (n-1)d}$

Corollary 2 . The sum of arithmetic sequence (T_n) 1,2,3,4, 5, ….n is given by

$$T_n = \frac{n(n+1)}{2}$$

Proof: The corollary easily follows by Theorem 1 with $A_1=1$ and $A_n = n$.

Problem 11:
The first term of an arithmetic sequence is equal to 10 and the common difference is equal to 5. Find the sum of 50 th term

Solution. Using $A_n = A_1 + (n - 1)d$, we have

$$A_{50} = 10 + (50 - 10)5$$

$$= 210$$

Now $S_{50} = \dfrac{50(10+210)}{2}$

$=5550$

Therefore the sum of 50 th term is 5550.

Problem 12.

Suppose that you play black jack at Las Vegas on June 1 and lose $1,000. Tomorrow you bet and lose $15 less. Each day you lose $15 less that your previous loss. What will your total losses be for the 30 days of June?

Solution

This is an arithmetic series problem. Note that

$A_1 = 1000$ and $d = -15$

We can calculate

$A_{30} = 1000 - 15(30 - 1) = 565$

Now we use the formula

$S_{30} = 30/2\ (1000 + 565) = 23{,}475$

You will lose a total of $23,475 during June.

Theorem 2. The sum S_n of the first n terms of a G.S. sequence is given by

$$S_n = \frac{G_1(r^{n+1}-1)}{r-1}$$

Proof: *Note the following:*

(1) $s_n = \sum_{k=0}^{n} G_1 r^k = G_1(1+r+r^2+r^3+\ldots+r^n)$

(2) $r\, s_n = G_1(r+r^2+r^3+\ldots+r^{n+1})$

Then (1) -(2) gives us

$rs_n - s_n = G_1(r^{n+1}-1)$

$\Rightarrow s_n(1-r) = G_1(r^{n+1}-1)$

$$\Rightarrow s_n = \frac{G_1(r^n-1)}{r-1} \quad (r \neq 0)$$

Corollary 3. The sum of the first *n* terms of the geometric sequence $1, 2, 2^2, 2^3, 2^4, \ldots 2^{n-1}$ is 2^n-1.

Proof: The corollary easily follows by Theorem 2.

Corollary 4 The sum of the first *n* terms of the geometric sequence **$1, 9, 9^2, 9^3, \ldots 9^n$ is** $\frac{9^{n+1}-1}{8}$

Proof: The corollary easily follows by Theorem 2

Problem 13 (Is this crazy mathematics ?)

A young 12 grade girl approaches a businessman and asks for a job. The businessman indicates that he's not sure he needs any help. The student suggests: "I'll work the first day for just a penny. On the second day I'll work for only 2 pennies. On the third day I'll work for 4 pennies ... etc. All I ask is that you guarantee me 4 weeks ... 20 days ... of work."

Should the businessman accept the girl's offer?

Solution: This is a geometric sequence. The sequence for pay is as follows:

\$.01, \$.02, \$.04, \$.08, \$.16, \$.32, \$.64 ... twenty total terms ... multiplier is 2. The last term $G_{20} = .01 * 2^{19}$ = \$ 5,242.88

The sum S_{20} of all twenty days is = [.01 * (1 - 2^{20})] / [1 - 2] = \$ 10,485.75

Therefore the businessman may not hire the girl according to this mathematics.

Theorem 3: You can not have a sequence of more than 2 terms (other than the trivial case of the constant sequence) that is both an A.S. and a G.S.

Proof: Consider 3 consecutive terms of a G.S. Let them be a, ar and ar^2. If these 3 terms also formed an A.S., the difference between the first 2 terms would equal the difference between the next two terms. In other words we would have

$$ar - a = ar^2 - ar$$
$$\Rightarrow a(r-1) = ar(r-1)$$
$$\Rightarrow a=0,\ r=1$$

So, no other sequence of 3 or more terms can be both a G.S. and an A.S.

Remark 1. Next we investigate beautiful connections that exist between these sequences and the seemingly unrelated mathematical territories of perfect and triangular numbers.

Theorem 4 **The sum** of the first n terms of the geometric sequence $1, 2, 2^2, 2^3, 2^4, \ldots 2^{n-1}$ multiplied by times the nth term is a triangular number.

Proof: Note that by Corollary 3, the sum equal to 2^n -1 and the nth term is 2^{n-1}. Hence, we must show that T= $2^{n-1}\left(2^n - 1\right)$ is
A triangular number. We have

$$T = 2^{n-1}\left(2^n - 1\right) = \frac{2^n(2^n-1)}{2} = \frac{m(m+1)}{2}, \text{ where } m = 2^n - 1 .$$

Theorem 5 If the sum of the first n terms of the geometric sequence $1, 2, 2^2, 2^3, 2^4, \ldots 2^{n-1}$ is a prime number, then this sum multiplied by the nth term is a perfect number

Proof: The sum equal to 2^n -1 and the nth term is 2^{n-1}. Hence, we must show that M= $2^{n-1}\left(2^n - 1\right)$ is perfect (equals to the sum of its proper factors) .We will find all the proper factors of $2^{n-1}\left(2^n - 1\right)$, and add them. Since 2^n-1 is prime, let $p = 2^n$-1 Then M = p(2^n-1)

Let us list all factors of 2^n-1 and other proper factors of m as follows .

Factors of 2^n-1	Other Proper Factors of M
1	p
2	$2p$
2^2	2^2p
2^3	2^3p
:	:
:	:
2^n-1	$2^{n-2}\,p$

Adding the first column, we get:

$$1+2+2^2+2^3...+2^{n-3}+2^{n-2}+2^{n-1}$$

$$=2^n-1$$

$$=p$$

Adding the second column, we get:

$$p+2p+2^2p+2^3p\ldots+2^{k-4}p+2^{k-3}p+2^{k-2}p$$

$$=p\left(1+2+2^2+\ldots+2^{k-2}\right)$$

$$=\left(2^{k-1}-1\right)p$$

Now adding the two columns together, we get:

$$p+p\left(2^{k-1}-1\right)$$

$$=p\left(1+2^{k-1}-1\right)$$

$$=p\left(2^{k-1}\right)$$

$$=M$$

Hence, M is a perfect number.

Theorem 6: Then partial sum of arithmetic sequence $1,2,3,4...(2^k-1).$ is a triangular number.

Proof: Let T be the sum of this arithmetic sequence. Note that:

$$
\begin{aligned}
T &= 1+2+3+4 \ldots + (2^k - 1) \\
&= \frac{\left(2^k - 1 + 1\right)\left(2^k - 1\right)}{2} \\
&= \frac{2^k\left(2^k - 1\right)}{2} \\
&= 2^{k-1}\left(2^k - 1\right) = \frac{m(m+1)}{2}, \ \textit{where } m = \left(2^k - 1\right).
\end{aligned}
$$

Hence, T is a triangular number.

Theorem 7: If the 2^k-1 is a prime number, then the partial sum of arithmetic sequence $1, 2, 3, 4 \ldots (2^k - 1)$. is a perfect number.

Proof: Let n be the sum of this arithmetic sequence. Note that:

$$
\begin{aligned}
n &= 1+2+3+4 \ldots + (2^k - 1) \\
&= \frac{\left(2^k - 1 + 1\right)\left(2^k - 1\right)}{2} \\
&= \frac{2^k\left(2^k - 1\right)}{2} \\
&= 2^{k-1}\left(2^k - 1\right)
\end{aligned}
$$

$\Rightarrow$ n is a perfect number by Theorem 4.

Theorem 8. The partial sum of the geometric sequence $2^{k-1}, 2^k, 2^{k+1}, \ldots 2^{2k-2}$ is a triangular number (k is an integer).

Proof: Note that T = $2^{k-1} + 2^k + \ldots\ldots + 2^{2k-2} = 2^{k-1}(1 + 2 + 2^2 + \ldots\ldots + 2^{k-1})$

$$= 2^{k-1}(2^k - 1) \quad \text{(by Corollary 3)}$$

$$= \frac{2^k(2^k - 1)}{2}$$

$$= \frac{m(m+1)}{2}, \text{ where } m = 2^k - 1.$$

Hence, T is a triangular number [2]

Theorem 9. For any natural number n, the partial sum of the geometric sequence **1 , 9, 9^2 , 9^3 , ... 9^n** is a triangular number[2].

Proof: By Corollary 4, we have $S_n = \frac{9^{n+1}-1}{8}$.

$$\Rightarrow 8S_n = 9^{n+1} - 1$$

$$\Rightarrow 8S_n + 1 = 9^{n+1} = (3^{n+1})^2$$

Since $8S_n + 1$ is odd and a perfect square, we have

$$8S_n + 1 = (2m+1)^2 = 4m^2+4m+1$$

$$\Rightarrow 8S_n = 4(m^2+m)$$

$$\Rightarrow S_n = \frac{m(m+1)}{2}$$

Hence, S_n is a triangular number [2]

5. Sum of Infinite Geometric Sequence

Let 1, r, r^2, r^3, r^4 …r^n … be an infinite geometric sequence.

Note that the following:

In Theorem 2 if G_1 =1 and $|r|<1$, we have $\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$

This is the sum of Infinite Geometric Sequence for $|r|<1$.

Some Problem Solving Techniques using Formula for Sum of Infinite Geometric Sequence

Problem 14:

Find the exact sum of : $\sum_{k=1}^{\infty}\left(\frac{6^{-k}+4^{-k}}{3^{-k}}\right)$

Solution:

Note that $\sum_{k=1}^{\infty}\left(\frac{6^{-k}+4^{-k}}{3^{-k}}\right)$

$$= \sum_{k=1}^{\infty}\left(\frac{3^{k}}{6^{k}}+\frac{3^{k}}{4^{k}}\right)$$

$$=\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^{k}+\sum_{k=1}^{\infty}\left(\frac{3}{4}\right)^{k}$$

$$=\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^{k+1}+\sum_{k=0}^{\infty}\left(\frac{3}{4}\right)^{k+1}$$

$$=\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^{k}+\frac{3}{4}\sum_{k=0}^{\infty}\left(\frac{3}{4}\right)^{k}$$

$$= \frac{1}{2}\left(\frac{1}{1-\frac{1}{2}}\right)+\frac{3}{4}\left(\frac{1}{1-\frac{3}{4}}\right)$$

=1+3=4

Problem 15: Find the exact sum of: $\sum_{k=2}^{\infty}(-1)^{k}(2)^{-2k}$

Solution: $\sum_{k=2}^{\infty}(-1)^{k}(2)^{-2k} = \sum_{k=2}^{\infty}(-1)^{k}\left(\frac{1}{4}\right)^{k}$

$$=\sum_{k=2}^{\infty}\left(\frac{-1}{4}\right)^{k}$$

$$=\sum_{k=0}^{\infty}\left(-\frac{1}{4}\right)^{k+2}$$

$$=\sum_{k=0}^{\infty}\left(-\frac{1}{4}\right)^{k}\left(-\frac{1}{4}\right)^{2}$$

$$=\frac{1}{16}\sum_{k=0}^{\infty}\left(-\frac{1}{4}\right)^{k}$$

$$=\frac{1}{16}\left(\frac{1}{1+\frac{1}{4}}\right)$$

$$=\frac{1}{16}\left(\frac{4}{5}\right)$$

$$=\frac{1}{20}$$

Problem 16: Find the sum of: $46+27+9+6+4+\frac{8}{3}+\ldots$

Solution: *Note that* $46+27+9+6+4+\frac{8}{3}+\ldots$

$$= 73+9\left(1+\frac{2}{3}+\left(\frac{2}{3}\right)^{2}+\left(\frac{2}{3}\right)^{3}+\ldots\right)$$

$$= 73+9\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^{k}$$

$$=73+9(3)=100$$

Problem 17: Show that: $\frac{17+15+48+24+12+6\ldots}{5+17+20+16+8+4+2\ldots}=2$

Solution

$$\frac{17+15+48+24+12+6...}{16+8+4+2...}$$

$$=\frac{17+15+48\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}\right)..}{5+17+20+16\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}\right)...}$$

$$=\frac{32+48\left(\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^{k}\right)}{42+16\left(\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^{k}\right)}$$

$$=\frac{32+48(2)}{42+16(2)}$$

$$=\frac{32+96}{42+32}$$

$$=\frac{148}{74}$$

$$=2$$

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