The Comparison of the convergence rate with different preconditioners for Linear Systems

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Abstract

In this paper, the preconditioned Gauss-Seidel iterative methods are proposed with different preconditioners. The comparison theorem is obtained under the different preconditioners when the coefficient matrix A of linear system is a nonsingular M− matrix. This generalizes the result in [1]. Numerical example are given to illustrate our theoretical result.

Keywords: Gauss-Seidel iterative, spectral radius, M-matrix, preconditioner

I Introduction

We consider the linear system of n equations

$$Ax = b \qquad (1)$$

Where $A = (a_{ij}) \in R^{n\times n}$ and $b \in R^n$ are given and $x \in R^n$ is unknown.

Assume that

$$A = M - N$$

Where M is nonsingular. Then the basic iterative method for solving (1) can be expressed in the form

$$x^{(k+1)} = M^{-1}Nx^{(k)} + M^{-1}b, k = 0,1,\cdots$$

Where $x^{(0)}$ is an initial vector. As it is well known, the above iterative process is convergent to the unique solution $x = A^{-1}b$ for each initial value $x^{(0)}$ if and only if the spectral radius of the iteration matrix $M^{-1}N$ satisfies $\rho(M^{-1}N) < 1$.

For simplicity, we let $A = I - L - U$, where I is the identity matrix, L and U are strictly lower and strictly upper triangular matrices, respectively. Then the iteration matrix of the

Gauss-Seidel iterative method for solving the linear system (1) is

$$T = (I - L)^{-1}U \qquad (2)$$

In order to accelerate the convergence of iterative method for solving the linear system (1), the original system (1) is transformed into the following preconditioned linear system

$$PAx = Pb \qquad (3)$$

where $P \in R^{n \times n}$ is nonsingular and called a preconditioner. Then the corresponding basic iterative method is given in general by

$$x^{(k+1)} = M_p^{-1} N_p x^{(k)} + M_p^{-1} Pb, k = 0,1,2\cdots$$

where $PA = M_p - N_p$ is a splitting of PA and M_p is nonsingular. Similar to the original system (1), we call the basic iterative methods corresponding to the preconditioned system the preconditioned iterative methods, such as the preconditioned Gauss-Seidel method and preconditioned AOR iterative method.

In [1]-[9], some different preconditioners have been proposed by several authors. In [1], the author presented preconditioned Gauss-Seidel method for linear systems and compared the convergence rate by using different preconditioners.

In this paper, we propose the new preconditioned Gauss-Seidel with the preconditoners P_1 and P_2, respectively. Furthermore, we compare the convergence rate of preconditioned Gauss-Seidel with the preconditoners P_1 and P_2.

The preconditioner P_1 is of the form $P_1 = I + R_\alpha + U$, where

$$R_\alpha = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ -\alpha_1 a_{n1} & -\alpha_2 a_{n2} & \cdots & -\alpha_{n-1} a_{nn-1} & 0 \end{pmatrix}$$

and $\alpha_i (i=1,2,\cdots,n-1)$ are real numbers. If $\alpha_i = 1(i=1,2,\cdots,n-1)$, the R_α becomes R in [1].

The preconditioner P_2 is of the form $P_2 = I + R_\alpha + S$, where

$$S = \begin{pmatrix} 0 & -a_{12} & 0 & \cdots & 0 \\ 0 & 0 & -a_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & -a_{n-1n} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

If $\alpha_i = 1(i=1,2,\cdots,n-1)$, the preconditioners P_1 and P_2 become the preconditioners P_{RU} and P_{SR}, respectively.

For convenience, some notations, definitions, lemmas and the theorems that will be used in the following parts are given below.

II Preliminaries

In this paper, $\rho(\cdot)$ denotes the spectral radius of a matrix.

Definition 2.1([10]). For $A=(a_{ij})$, $B=(b_{ij})\in R^{n\times n}$, we write $A \ge B$, if $a_{ij} \ge b_{ij}$ holds for all $i,j=1,2,\cdots,n$. Calling A nonnegative matrix if $A \ge 0(\mathrm{a_{ij}} \ge 0, i,j=1,2,\cdots,n)$.

Definition 2.2([11]). A matrix A is a L-matrix if $a_{ii} \ge 0.i=1,2,\cdots,n$ and $a_{ij} \le 0$ for all $i,j=1,2,\cdots,n,\ \ i\ne j$. A nonsingular L-matrix A is a nonsingular M-matrix if $A^{-1} \ge 0$.

Lemma 2.1([10]). Let A be a nonnegative $n\times n$ nonzero matrix. Then

(a) $\rho(A)$, the spectral radius of A, is an eigenvalue;

(b) A has a nonnegative eigenvector corresponding to $\rho(A)$;

(c) $\rho(A)$ is a simple eigenvalue of A;

(d) $\rho(A)$ increases when any entry of A increases.

Definition 2.3([10]). For $n\times n$ real matrices A, M and N, $A=M-N$ is a regular splitting of the matrix A if M is nonsingular with $M^{-1} \ge 0$ and $N \ge 0$. Similarly,

$A=M-N$ is a weak regular splitting of the matrix A if M is nonsingular with $M^{-1} \ge 0$

and $M^{-1}N \geq 0$.

Lemma 2.2([2]). Let A be a nonnegative matrix. Then

(1) If $\alpha x \leq Ax$ for some nonnegative vector $x, x \neq 0$, then $\alpha \leq \rho(A)$.

(2) If $Ax \leq \beta x$ for some positive vector x, then $\rho(A) \leq \beta$. Moreover, if A is irreducible and if $0 \neq \alpha x \leq Ax \leq \beta x$, $\alpha x \neq Ax$, $Ax \neq \beta x$ for some nonnegative vector x, then $\alpha < \rho(A) < \beta$ and x is a positive vector.

Lemma 2.3([1]). Suppose that $A_1 = M_1 - N_1$ and $A_2 = M_2 - N_2$ are weak regular splitting of the monotone matrices A_1 and A_2, respectively, such that $M_2^{-1} \geq M_1^{-1}$. If there exists a positive vector x such that $0 \leq A_1 x \leq A_2 x$. Then, for the monotonic norm associated with x,

$$\|M_2^{-1}N_2\|_x \leq \|M_1^{-1}N_1\|_x$$

In particular, if $M_1^{-1}N_1$ has a positive Perron vector, then

$$\rho(M_2^{-1}N_2) \leq \rho(M_1^{-1}N_1)$$

III Preconditioned Gauss-Seidel iterative method and comparison theorem

For the linear system (1), we consider its preconditioned from

$$A_1 x = P_1 A x = P_1 b \qquad (4)$$

where $P_1 = I + R_\alpha + U$.

Now, we express the coefficient matrix of (4) as

$$\begin{aligned} A_1 &= P_1 A = (I + R_\alpha + U)(I - L - U) \\ &= I - L - U + R_\alpha - R_\alpha L - R_\alpha U + U - UL - U^2 \\ &= I - D_0 - D_1 - (L - R_\alpha + R_\alpha L + E_1 + E_0) - (F_0 + U^2) \\ &= M_{UR_\alpha} - N_{UR_\alpha} \end{aligned}$$

where $UL = D_0 + E_0 + F_0$ and $R_\alpha U = D_1 + E_1$. D_0, E_0 and F_0 are diagonal, strictly lower and upper triangular parts of UL, respectively. D_1 and E_1 are diagonal and strictly lower triangular parts of $R_\alpha U$.

Suppose that $M_{UR_\alpha} = I - D_0 - D_1 - (L - R_\alpha + R_\alpha L + E_1 + E_0)$ (5)

$N_{UR_\alpha} = F_0 + U^2$ (6)

Then the perconditioned Gauss-Seidel iteration matrix with the preconditioner P_1

$$T_{UR_\alpha} = M_{UR_\alpha}^{-1} N_{UR_\alpha} = [(I - D_0 - D_1) - (L - R_\alpha + R_\alpha L + E_1 + E_0)]^{-1}(F_0 + U^2) \quad (7)$$

Similarly, we consider its preconditioned form

$$A_2 x = P_2 A x = P_2 b \quad (8)$$

where $P_2 = I + R_\alpha + S$.

We express the coefficient matrix of (8) as

$$\begin{aligned} A_2 &= P_2 A = (I + R_\alpha + S)(I - L - U) \\ &= I - L - U + R_\alpha - R_\alpha L - R_\alpha U + S - SL - SU \\ &= I - D_1 - D_2 - (L - R_\alpha + R_\alpha L + E_1 + E_2) - (U - S + SU) \\ &= M_{SR_\alpha} - N_{SR_\alpha} \end{aligned}$$

Where $SL = D_2 + E_2$, D_2 and E_2 are diagonal and strictly lower triangular parts of SL, respectively.

Suppose that

$M_{SR_\alpha} = I - D_1 - D_2 - (L - R_\alpha + R_\alpha L + E_1 + E_2)$ (9)

$N_{SR_\alpha} = U - S - SU$ (10)

Then the perconditioned Gauss-Seidel iteration matrix with the preconditioner P_2

$$T_{SR_\alpha} = M_{SR_\alpha}^{-1} N_{SR_\alpha} = [(I - D_1 - D_2) - (L - R_\alpha + R_\alpha L + E_1 + E_2)]^{-1}(U - S + SU) \quad (11)$$

Theorem 3.1 Let A_1 and A_2 be the coefficient matrices of linear system (4) and (8), respectively. $M_{UR_\alpha}, N_{UR_\alpha}, M_{SR_\alpha}$ and N_{SR_α} are defined by (5),(6),(9) and (10), respectively. Let A be a nonsingular M -matrix. Suppose that $0 \le \sum_{j=k+1}^{n} a_{kj} a_{jk} < 1$, $0 \le \sum_{i=1}^{n-1} \alpha_i a_{ni} a_{in} < 1$ and $0 \le \alpha_i \le 1$ for $i = 1, 2, \cdots, n-1$. Then $A_1 = M_{UR_\alpha} - N_{UR_\alpha}$ and $A_2 = M_{SR_\alpha} - N_{SR_\alpha}$ are weak regular splitting of A_1 and A_2, respectively.

Proof. First, we prove that $A_1 = M_{UR_\alpha} - N_{UR_\alpha}$ is weak regular splitting of A_1. Since A is nonsigular M-matrix, $0 \le \sum_{j=k+1}^{n} a_{kj}a_{jk} < 1$, $0 \le \sum_{i=1}^{n-1} \alpha_i a_{ni} a_{in} < 1$ and $0 \le \alpha_i \le 1$,

$$\begin{aligned}
M_{UR_\alpha}^{-1} &= [(I - D_0 - D_1) - (L - R_\alpha + R_\alpha L + E_1 + E_0)]^{-1} \\
&= [I - (I - D_0 - D_1)^{-1}(L - R_\alpha + R_\alpha L + E_1 + E_0)]^{-1}(I - D_0 - D_1)^{-1} \\
&= \{I + (I - D_0 - D_1)^{-1}(L - R_\alpha + R_\alpha L + E_1 + E_0) + [(I - D_0 - D_1)^{-1}(L - R_\alpha + R_\alpha L + E_1 + E_0)]^2 + \cdots\}(I - D_0 - D_1)^{-1} \\
&\ge 0
\end{aligned}$$

We know that $N_{UR_\alpha} = F_0 + U^2 \ge 0$. Therefor, $M_{UR_\alpha}^{-1} N_{UR_\alpha} \ge 0$. By Definition 2.3, we obtain that $A_1 = M_{UR_\alpha} - N_{UR_\alpha}$ is weak regular splitting of A_1.

Now, we will prove that $A_2 = M_{SR_\alpha} - N_{SR_\alpha}$ is weak regular splitting of A_2.

Since A is a nonsingular M-matrix, we have $0 \le a_{ii+1}a_{i+1i} < 1$ for $i = 1,2,\cdots,n-1$. According to $0 \le \sum_{i=1}^{n-1} \alpha_i a_{ni} a_{in} < 1$ and $0 \le \alpha_i \le 1$ ($i = 1,2,\cdots,n-1$), we obtain that

$$\begin{aligned}
M_{SR_\alpha}^{-1} &= [(I - D_1 - D_2) - (L - R_\alpha + R_\alpha L + E_1 + E_2)]^{-1} \\
&= [I - (I - D_1 - D_2)^{-1}(L - R_\alpha + R_\alpha L + E_1 + E_2)]^{-1}(I - D_1 - D_2)^{-1} \\
&= \{I + (I - D_1 - D_2)^{-1}(L - R_\alpha + R_\alpha L + E_1 + E_2) + [(I - D_1 - D_2)^{-1}(L - R_\alpha + R_\alpha L + E_1 + E_2)]^2 + \cdots\}(I - D_1 - D_2)^{-1} \\
&\ge 0
\end{aligned}$$

We know that $N_{SR_\alpha} = U - S + SU \ge 0$. By Definition 2.3, we obtain that $A_2 = M_{SR_\alpha} - N_{SR_\alpha}$ is weak regular splitting of A_2. This completes the proof.

Theorem 3.2 Let A_1 and A_2 be the coefficient matrices of linear system (4) and (8), respectively. M_{UR_α}, N_{UR_α}, M_{SR_α} and N_{SR_α} are defined by (5),(6),(9) and (10), respectively. Let A be a nonsingular M-matrix. Suppose that $0 \le \sum_{j=k+1}^{n} a_{kj}a_{jk} < 1$, $0 \le \sum_{i=1}^{n-1} \alpha_i a_{ni} a_{in} < 1$ and $0 \le \alpha_i \le 1$ for $i = 1,2,\cdots,n-1$. Then $\rho(M_{UR_\alpha}^{-1} N_{UR_\alpha}) \le \rho(M_{SR_\alpha}^{-1} N_{SR_\alpha})$.

Proof. For a positive vector x and A is a nonsingular M-matrix,

$A_1 x = (I + R_\alpha + U)Ax \geq (I + R_\alpha + S)Ax \geq 0$. We have

$$\begin{aligned} M_{SR_\alpha} - M_{UR_\alpha} &= (I - D_1 - D_2) - (L - R_\alpha + R_\alpha L + E_1 + E_2) \\ &- [(I - D_0 - D_1) - (L - R_\alpha + R_\alpha L + E_1 + E_0)] \\ &= (D_0 + E_0) - (D_2 + E_2) \\ &= (D_0 + E_0) - SL \geq 0 \end{aligned} \tag{12}$$

By Theorem 3.1, we know that $M_{UR_\alpha}^{-1} \geq 0$ and $M_{SR_\alpha}^{-1} \geq 0$. Pre-multiplying and post-multiplying (12) by $M_{UR_\alpha}^{-1}$ and $M_{SR_\alpha}^{-1}$, respectively, we have

$$M_{UR_\alpha}^{-1} - M_{SR_\alpha}^{-1} \geq 0$$

Thus, $M_{UR_\alpha}^{-1} \geq M_{SR_\alpha}^{-1}$. By Lemma 2.3 and Theorem 3.1, we obtain that

$$\rho\,(M_{UR_\alpha}^{-1} N_{UR_\alpha}) \leq \rho\,(M_{SR_\alpha}^{-1} N_{SR_\alpha})$$

This completes the proof.

Remark If $\alpha_i = 1$ for $i = 1, 2, \cdots, n-1$, Theorem 3.2 becomes the result of Theorem 4.3 in [1].

IV Numerical example

In this section, we give the following example to illustrate the results obtained in section 3.

Example The coefficient matrix A of (1) is given by

$$A = \begin{pmatrix} 1 & -0.2 & -0.3 & -0.1 & -0.2 \\ -0.1 & 1 & -0.1 & -0.3 & -0.1 \\ -0.2 & -0.1 & 1 & -0.1 & -0.2 \\ -0.2 & -0.1 & -0.1 & 1 & -0.3 \\ -0.1 & -0.2 & -0.2 & -0.1 & 1 \end{pmatrix}$$

We see that A satisfies the condition of Theorem 3.1 and Theorem 3.2.

If $\alpha_1 = 0.1$, $\alpha_2 = 0.2$, $\alpha_3 = 0.2$, $\alpha_4 = 1$, we denote the spectral radius of the preconditioned Gauss-Seidel iterative matrix with the preconditioners P_1 and P_2 by $\rho\,(M_{U_1R_\alpha}^{-1} N_{U_1R_\alpha})$ and $\rho\,(M_{S_1R_\alpha}^{-1} N_{S_1R_\alpha})$, respectively.

If $\alpha_1 = 0.2$, $\alpha_2 = 0.5$, $\alpha_3 = 1$, $\alpha_4 = 0.2$, we denote the spectral radius of the preconditioned

Gauss-Seidel iterative matrix with the preconditioners P_1 and P_2 by $\rho(M_{U_2R_\alpha}^{-1}N_{U_2R_\alpha})$ and $\rho(M_{S_2R_\alpha}^{-1}N_{S_2R_\alpha})$, respectively.

If $\alpha_1=0.8$, $\alpha_2=0.2$, $\alpha_3=0.3$, $\alpha_4=0.5$, we denote the spectral radius of the preconditioned Gauss-Seidel iterative matrix with the preconditioners P_1 and P_2 by $\rho(M_{U_3R_\alpha}^{-1}N_{U_3R_\alpha})$ and $\rho(M_{S_3R_\alpha}^{-1}N_{S_3R_\alpha})$, respectively.

If $\alpha_1=0.1$, $\alpha_2=1$, $\alpha_3=1$, $\alpha_4=1$, we denote the spectral radius of the preconditioned Gauss-Seidel iterative matrix with the preconditioners P_1 and P_2 by $\rho(M_{U_4R_\alpha}^{-1}N_{U_4R_\alpha})$ and $\rho(M_{S_4R_\alpha}^{-1}N_{S_4R_\alpha})$, respectively.

If $\alpha_1=0.9$, $\alpha_2=0.4$, $\alpha_3=0.8$, $\alpha_4=0.5$, we denote the spectral radius of the preconditioned Gauss-Seidel iterative matrix with the preconditioners P_1 and P_2 by $\rho(M_{U_5R_\alpha}^{-1}N_{U_5R_\alpha})$ and $\rho(M_{S_5R_\alpha}^{-1}N_{S_5R_\alpha})$, respectively. Then we obtain the Table 1.

Table 1 The comparison of the spectral radius of preconditioned Gauss-Seidel iterative matrix with the preconditioners P_1 and P_2

i	$\rho(M_{U_iR_\alpha}^{-1}N_{U_iR_\alpha})$	$\rho(M_{S_iR_\alpha}^{-1}N_{S_iR_\alpha})$
$i=1$	0.1818	0.3313
$i=2$	0.1745	0.3110
$i=3$	0.1732	0.3137
$i=4$	0.1570	0.2724
$i=5$	0.1670	0.3002

From Table 1, we can see that $\rho(M_{UR_\alpha}^{-1}N_{UR_\alpha})\le\rho(M_{SR_\alpha}^{-1}N_{SR_\alpha})$.

Conjectures In this paper, the preconditioners P_1 and P_2 are generalized to the preconditioners with multi-parameters, the result may be correct.

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